Thermal Johnson Noise
Generated by a Resistor

REFERENCES

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- Kittel, *Thermal Physics*, pp. 98-102
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- Nyquist, *Phys. Rev.* 32, 110 (1928)

HISTORY

In 1926, experimental physicist John Johnson working in the physics division at Bell Labs was researching noise in electronic circuits. He discovered that there was an irreducible low level of noise in resistors whose power was proportional to temperature. Harry Nyquist, a theorist in that division, got interested in the phenomenon and developed an elegant explanation based on fundamental physics.

THEORY OF THERMAL JOHNSON NOISE

Thermal agitation of electrons in a resistor gives rise to random fluctuations in the voltage across its terminals, known as Johnson noise. In a narrow band of frequencies, Δf, the contribution to the mean-squared noise voltage from this thermal agitation is,

$$\left\langle V(t)^2 \right\rangle_{time} = 4Rk_B T \Delta f \tag{1}$$

where R is the resistance in ohms and T is the temperature in degrees Kelvin for the resistor, k_B is the Boltzmann constant (1.38×10^{-23} J/K).

This voltage is usually too small to be detected without amplification. If the resistor is connected across the input of a high-gain amplifier whose voltage gain as a function of frequency is $G(f)$, the mean square of the voltage output of the amplifier will be:

$$\left\langle V(t)^2 \right\rangle_{time} = 4Rk_B T \int_0^\infty [G(f)]^2 df + \left\langle V(t)_N^2 \right\rangle_{time} \tag{2}$$

where $\left\langle V(t)_N^2 \right\rangle_{time}$ is the output noise generated by the amplifier itself.

Thus by measuring $\langle V(t)^2 \rangle_{time}$ as a function of R and making a plot, one obtains $4k_B T \int_0^\infty [G(f)]^2 df$ from the slope, while the abscissa gives $\langle V(t)_N^2 \rangle_{time}$. But the amplifier gain $G(f)$ can be independently measured and the gain integral $\int_0^\infty [G(f)]^2 df$ evaluated. The slope will then give a value for the Boltzmann constant k_B.

Derivation of Eq. (1)

An electrical transmission line connected at one end to a resistor R and at the other end by an "equivalent" resistor R may be treated as a one- dimensional example of black body radiation.
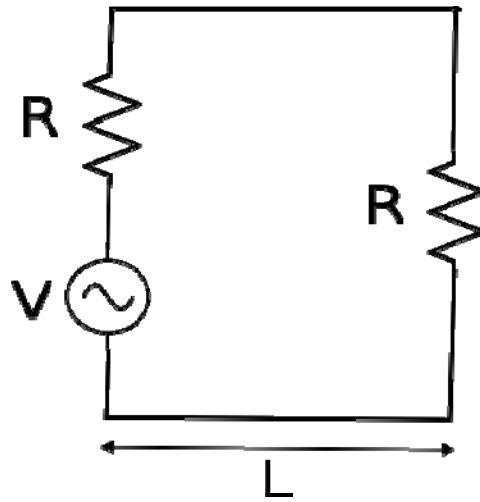

Figure 1. Two resistors (of equal resistance R) coupled via a transmission line L.

At finite temperature T, the resistor R generates a noise voltage $V(t)$ which will propagate down the line. If the characteristic impedance of the transmission line is made equal to R, the radiation incident on the "equivalent" resistor R from the first resistor R should be completely absorbed.

The permitted standing wave modes in the line have $\lambda = 2L/n$ and $f = (c/2L)n$, where $n = 1,2,3$, etc., and v is the wave velocity in the line. The separation of the modes in frequency is $v/2L$ and the number of modes between f and $f + \Delta f$ is

$$\sigma(f)\Delta f = (2L/c)\Delta f \qquad (3)$$

From the Planck distribution or the equipartition theorem, the mean thermal energy contained in each electromagnetic mode or photon state in the transmission line is:

$$\langle E(f)\rangle = \frac{hf}{e^{hf/k_BT}-1} \approx k_BT \quad \text{at low frequencies.} \tag{4}$$

From Eq. (3) and (4) find the electromagnetic energy $\langle E(f)\rangle\sigma(f)\Delta f$ in a frequency interval Δf. One half of this energy is generated by the first resistor of R and propagating towards the "equivalent" resistor R. Knowing the propagation time from the generating resistor to the absorbing resistor $\Delta t = L/c$, show that the absorbed power by the "equivalent" resistor R equals

$$P(f)\Delta f = k_BT\Delta f. \tag{5}$$

In thermal equilibrium, this power is simply the ohmic heating generated by a noise voltage source $V(t)$ from the first resistor. Since $V(t)$ is terminated by the absorbing resistor R and has an "internal" resistance R (the first resistor), it produces a current $I = V/(2R)$ in the line. Hence the power absorbed by the "equivalent" resistor R over the frequency interval Δf can also be calculated as

$$I^2R = \left(\frac{V}{2R}\right)^2 R = \frac{V^2}{4R} = \frac{V^2(f)\Delta f}{4R} \tag{6}$$

By equating $P(f)\Delta f = k_BT\Delta f$ to $V^2\Delta f/R$, show that

$$V^2(f)\Delta f = 4k_BTR\Delta f \tag{7}$$

and

$$\langle V(t)^2\rangle_{time} = 4k_BTR\Delta f \tag{8}$$

This is known as Nyquist's theorem as shown in Eq. (1). The power spectral density (noise power per unit frequency) is independent of frequency. Most other noise sources in nature have a f^{-1} to f^{-2} spectrum.

   Question: what is the integrated power of this Johnson noise over all frequencies? [i.e., why can't a single resistor supply the world's energy needs?]

Eq. (1) is interesting: the left hand side describes random fluctuations of a lossy system in thermodynamic equilibrium. i.e. it is an equilibrium property; both the electrons and the lattice are in equilibrium at some temperature T. However, the right hand side of the equation refers explicitly to a non-equilibrium property of the same lossy system: the resistance R, which is measured by applying a voltage (taking the system out of equilibrium) and measuring the current of electrons scattering through the lossy system. **Johnson noise is an example of a broader fundamental principle in nature called the "*fluctuation-dissipation theorem*." It relates the non-equilibrium dissipation in a system to its spontaneous fluctuations in equilibrium. See accompanying papers by Callen.** A similar phenomenon is Brownian motion.