Sampling Theory and Spline Interpolation

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Abstract

In this module, Shannon's classical sampling theory is compared to digital to analog signal reconstruction using spline interpolation. In the spline method, the signal is reconstructed using sample-weighted cardinal splines as opposed to sample-weighted sinc functions.

Sampling Theory and Spline Interpolation

1 Shannon's sampling theory

Shannon's sampling theory[1] tells us that if we have a bandlimited signal[2] ($s(x)$) that has been sampled at the Nyquist rate[3], then the signal can be reconstructed from its samples ($s[k]$) with the following relation:

$$s(x) = \sum_{k \in \mathbb{Z}} (s[k] \operatorname{sinc}(x-k)) \tag{1}$$

This relation is frequently used in digital to analog converter. There are several desirable properties of the sinc function[4] that make this strategy effective. First of all, the sinc function vanishes at all integers except at the origin. Secondly, $\operatorname{sinc}(0) = 1$. As a result, if T is the sampling frequency, then $s(xT) = s[k]$. The reconstruction of the sequence of samples $[1, 2, 3, 3, 1.5, 0, 1, 4]$ can be seen in Figure 1.

The disadvantage of this approach is that it depends on the initial assumption that the signal is bandlimited, but frequently we rely on only a finite number of samples, which cannot completely describe a bandlimited signal. As a result, we can only find an approximate estimate of the signal $s(x)$.

*http://creativecommons.org/licenses/by/1.0
[1]http://cnx.rice.edu/content/m11044/latest/
[2]http://cnx.rice.edu/content/m10400/latest/
[3]http://cnx.rice.edu/content/m0090/latest/
[4]http://cnx.rice.edu/content/m10790/latest/

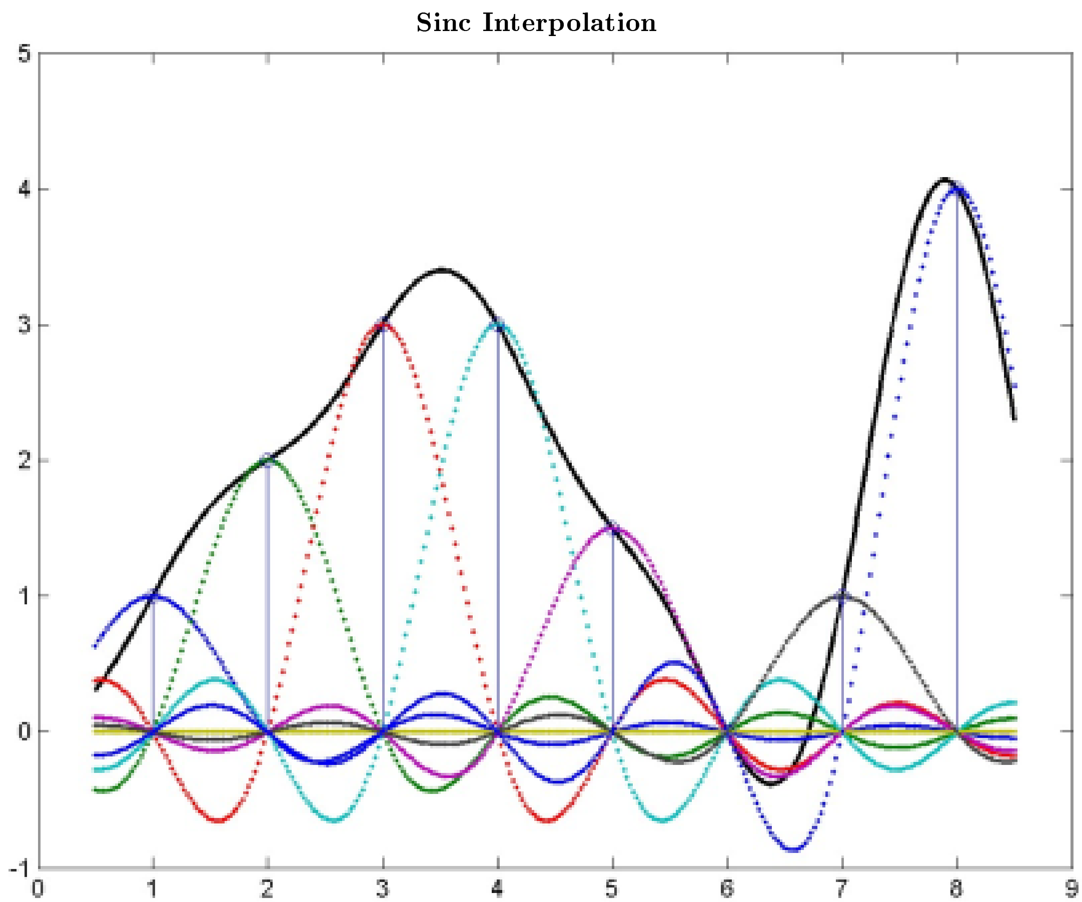

Figure 1: Bandlimited continuous signals can be reconstructed from their samples using a linear combination of sinc functions, where the sinc functions are weighted by the sample values.

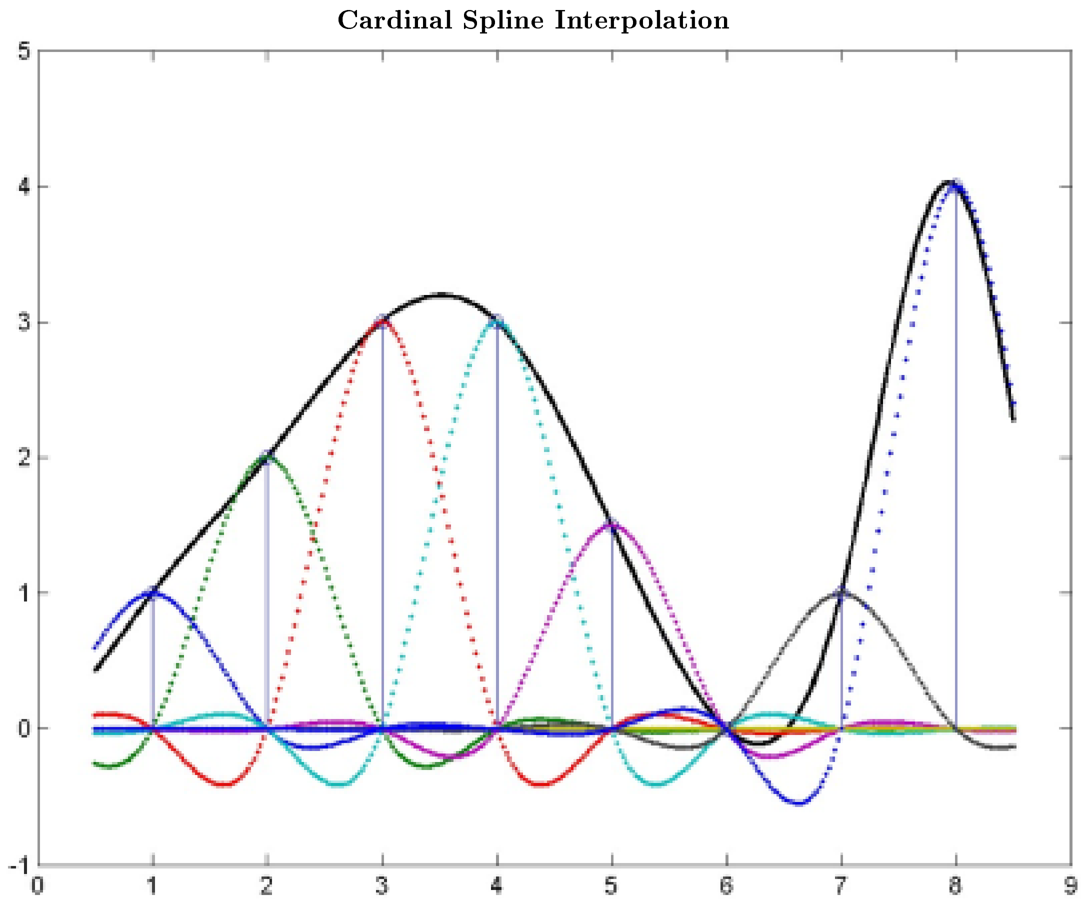

Figure 2: Bandlimited continuous signals can be reconstructed from their samples using a linear combination of cardinal splines, where the spline functions are weighted by the sample values.

2 Signal Reconstruction with Cardinal Splines

As described above, having only a finite number of samples leads to inaccuracies in estimating $s(x)$. Using cardinal splines[5] instead of sinc functions can lessen the magnitude of the errors. The n^{th} cardinal spline, η^n, gives piecewise polynomial interpolation with order n polynomials. Like the sinc function, each cardinal spline vanishes at all integers except the origin, and $\eta^n(0) = 1$. Furthermore $\underset{n\to\infty}{lim}\, \eta^n(x) = \mathrm{sinc}(x)$. This means that cardinal splines can be used for signal reconstruction from samples just as sinc functions are used. Specifically,

$$s(x) = \sum_{k\in\mathbb{Z}} \left(s[k]\, \eta^n(x-k)\right) \quad (2)$$

The reconstruction of the sequence of samples $[1, 2, 3, 3, 1.5, 0, 1, 4]$ can be seen in Figure 2.

From images Figure 1 and Figure 2, it may appear that the spline interpolation is

[5]http://cnx.rice.edu/content/m11127/latest/

smoother than the sinc interpolation. This is because the support of the cardinal splines is more compact than that of the sinc function. In fact, to compute the value of $s(x)$ (when $s(x)$ is a polynomial signal) with an error of less than 1%, one would need $O(100)$ sinc functions, but just $n+1$ B-splines for exact evaluation.