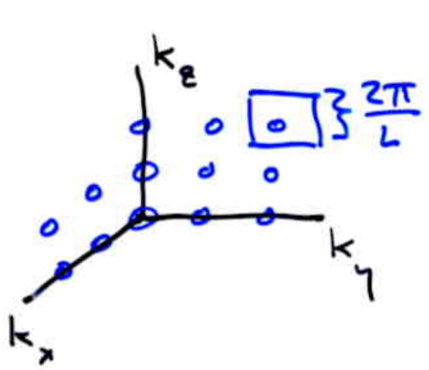
Black Body Radiation (counting modes)

Consider standing waves in cubic box of side L. Compute number of modes per unit volume $G(\nu)d\nu$ between frequency ν and $\nu + d\nu$.

Use periodic boundary conditions $E(x,y,z) = E(x+L, y, z)$ etc. Separate variables. Then $E(x) = E_0 e^{ik_x L}$ where $k_x = 2\pi n/L \quad n = 0, \pm 1, \pm 2 \ldots$

Modes are points on a cubic Lattice in 3-d k-space. Each point represents volume $(2\pi/L)^3$



Number of points $N(k)$ in spherical shell of radius $|k|$ and thickness $|dk|$:

$$N(k)\,dk = \frac{4\pi k^2 dk}{(2\pi/L)^3} \begin{array}{l} \leftarrow \text{volume of shell} \\ \leftarrow \text{volume per point} \end{array}$$

where $|k| = 2\pi \nu/c$

Change variables: $N(\nu)d\nu = N(k)\,dk = \frac{4\pi\nu^2 L^3}{c^3} d\nu$

Divide by volume L^3 and multiply $\times 2$ for polarization. $\rightarrow$ Number of modes of two polarizations between ν and $\nu + d\nu$ per unit volume:

$$G(\nu)\,d\nu = \frac{8\pi\nu^2}{c^3} d\nu$$

Energy density (i.e. per unit volume) $u(\nu)\,d\nu$

$$u(\nu)\,d\nu = G(\nu)\,d\nu \times \text{energy per mode.}$$

$$\text{Energy per mode} = \begin{cases} kT & (\text{Rayleigh-Jeans}) \\ \dfrac{h\nu}{e^{h\nu/kT}-1} & (\text{Planck}) \end{cases}$$

So, Planck spectral energy density in box:

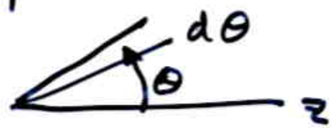
$$u(\nu)\,d\nu = \frac{8\pi\nu^2 d\nu \; h\nu}{c^3\left(e^{h\nu/kT}-1\right)}$$

This energy moves in all directions (4π solid angle). The fraction which occupies solid angle $d\Omega$ is $d\Omega/4\pi$. So, power per unit area radiated into solid angle $d\Omega$:

$$\frac{P(\nu)\,d\nu\,d\Omega}{A} = \left(\frac{d\Omega\,c}{4\pi}\right)u(\nu)\,d\nu$$

Integrate over angle to obtain total power per unit area in forward direction. In spherical coords:

$$d\Omega = \sin\theta\,d\theta\,d\phi$$



$$P(\nu)\,d\nu \int \cos\theta\,d\Omega = P(\nu)\,d\nu \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{2\pi} \cos\theta\,\sin\theta\,d\theta}_{1/2} = \pi\,P(\nu)\,d\nu$$

projected area is $A\cos\theta$

Result is spectral brightness (or spectral radiance)

$$B(\nu)\,d\nu = \frac{\text{Total } P(\nu)\,d\nu}{A} = \frac{c\,u(\nu)}{4}\,d\nu$$

Now integrate over all frequencies to get total brightness (radiance power per unit area, all angles, all frequencies):

$$B(T) = \frac{2\pi k^4 T^4}{c^2 h^3}\int_0^\infty \frac{x^3\,dx}{e^x - 1} = \sigma T^4$$

$$\sigma = \frac{2\pi^5 k^4}{15 h^3 c^2} = 5.67\times 10^{-8}\ \mathrm{W/m^2K^4}$$ Stefan-Boltzmann constant

Planck derivation gives spectral brightness:

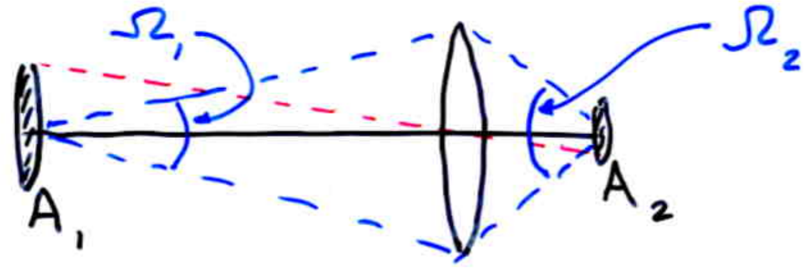
$$B(\nu,T)\,d\nu = \frac{2h\nu^3\,d\nu}{c^2\left(e^{h\nu/kT}-1\right)}\ \mathrm{W/m^2\,sr} = \frac{\text{Power}}{\text{Throughput}} = \frac{P_\nu\,d\nu}{A\Omega}$$

$A\Omega \equiv$ throughput, or "etendue".

It is an invariant through an optical system!

Using these ideas, here is a simpler and physically more instructive derivation of $B(\nu,T)$:

Consider a generalized optical system



For geometrical optics limit, you can prove $A\Omega$ is invariant simply by ray-trace.

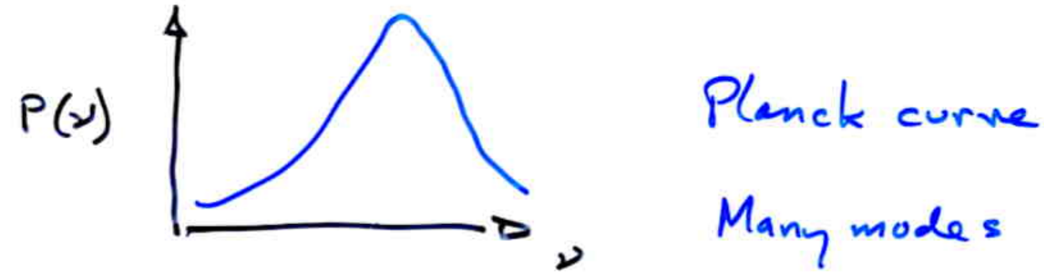
In the diffraction limit you can prove the invariance of $A\Omega$ by requiring the 2^{nd} Law of thermodynamics!

Make A_1 and A_2 into black bodies at temp T. Require power flow to left = power flow to right (2^{nd} law says a passive element like the lens cannot transfer energy in one direction).

"Antenna theorem": $A\Omega = \lambda^2 = c^2/\nu^2$ for single mode.

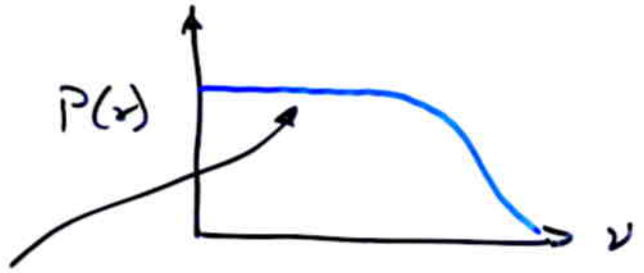
$\therefore$ Number of modes in a throughput $A\Omega$ is $A\Omega/c^2/\nu^2$ $\times 2$ for polarization. Since the power per mode $P_\nu d\nu = h\nu\, d\nu/(e^{h\nu/kT}-1)$, we get above result.

Plot $P_\nu d\nu$ holding $A\Omega$ constant:



However, some optical or microwave systems are limited to a single mode.

In that case, we get:



At low frequencies $P_\nu = kT$

So any power can be expressed as an effective temperature ("antenna temperature").

Photon noise from thermal sources

If photons arrive at random, from Poisson statistics the mean square fluctuation in the number of photons arriving in a time $\tau = 1\,\mathrm{sec}$ is equal to the number of arriving photons:

$$\overline{\Delta n^2} = \bar{n} \qquad \frac{\text{sig}}{\text{noise}} \sim n^{1/2}$$

As in electron shot noise [lecture 2] we write this variance as

$$S_n \, \Delta f_{eff}$$

where $\Delta f_{eff} = 1/2\tau$, so the spectral density is

$$S_n = 2\,\bar{n}$$

This white noise has been verified in many experiments with photodetectors, from visible light to γ-rays.

In a wide-band detector which measures power,

$$\bar{n} = P_\nu / h\nu$$

but not all photons are equal! For an experiment with wide IR bandwidth (like a bolometer) we must add the effects of photons of different energy

Then $$S_P = \frac{P_n^2}{\Delta\nu} = 2\int P_\nu \, h\nu \, d\nu$$

A more rigorous description would include photon clumping, since they are Bosons. So there is a "Bose" correction to the simple Poisson noise model:

$$\overline{\Delta n^2} = \bar{n}\left[1 + \frac{\varepsilon \tau \eta}{e^{h\nu/kT} - 1}\right]$$

where ε = emissivity of source, τ = transmission of optical system, and η = detector quantum efficiency. This Bose correction becomes important only for low frequencies and high efficiency (i.e. radio). [Note clumping adds extra shot noise]

Quantum Efficiency (QE)

A flux φ of photons is absorbed in a detector with absorption coefficient $a(\lambda)$ [units cm^{-1}]:

$$\varphi = \varphi_0 e^{-a l}$$

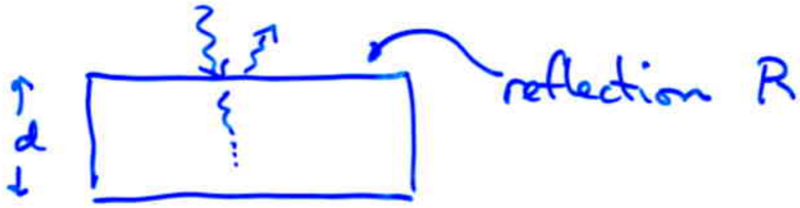
The QE $\equiv \eta$ is the flux absorbed in the detector divided by total incident flux. Two components:

1. fraction of incident photons which enter the detector and are absorbed within it

2. fraction which are reflected and not absorbed

Let detector thickness = d

Absorption factor $\eta_{abs} = 1 - e^{-a(\lambda)d}$



$$QE = \eta = (1-R)\,\eta_{abs}$$

QE is a function of wavelength, due to $\eta_{abs}(\lambda)$ and $R(\lambda)$. Semiconductor details.

Detective Quantum Efficiency (DQE)

S/N ratio in real systems is a result of many stages: detector, charge collection, amplifiers ...
Each stage can lose signal or add noise!

$$DQE = \frac{(S/N)^2_{OUT}}{(S/N)^2_{IN}}$$

In perfect system, DQE → QE of detector.

Noise Equivalent Power (NEP)

Rieke p50

A measure of overall system sensitivity is the NEP, defined as the signal power that gives an rms S/N = 1 for an output bandpass of 1 Hz.

The more sensitive the detector, the smaller the NEP. [Units $W / Hz^{1/2}$]

If P_s is the signal power and Δf the output integration electronics bandwidth

$$S/N = \frac{P_s}{NEP\,(\Delta f)^{1/2}} = \frac{P_s\,(2\,t_{int})^{1/2}}{NEP}$$

NEP can be measured by injecting a calibrated signal, integrating for time t, and measuring the S/N ratio:

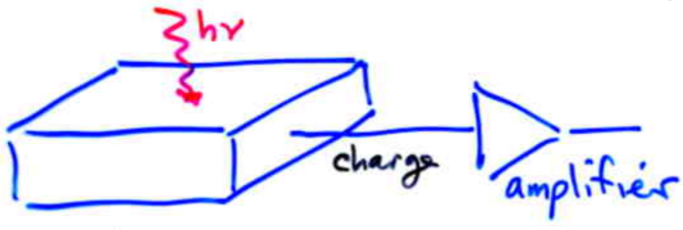
$$NEP = \frac{P_s\,(2\,t_{int})^{1/2}}{S/N}$$

Yet another definition:

$$D \equiv \text{Detectivity} = (NEP)^{-1}$$

Types of Detectors

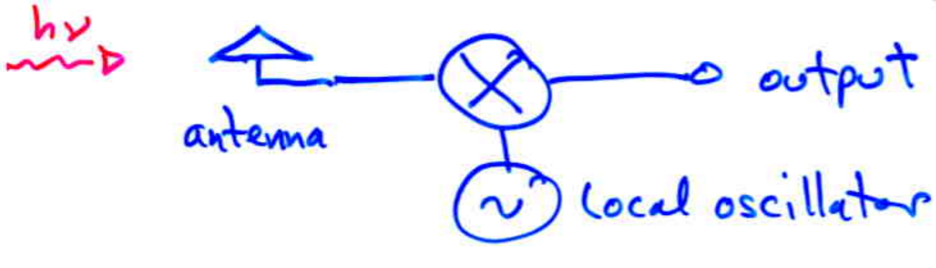
PHOTON DETECTORS Output response for <u>each</u> photon. Absorbed photon releases bound charge(s). Used over IR → X-ray range.



Examples: photoconductors, photodiodes and arrays of these. Some are <u>integrating</u>, so net signal charge overcomes internal noise.

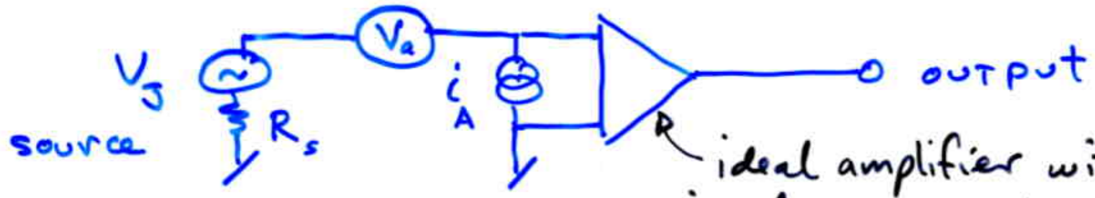
THERMAL DETECTORS Thermalize absorbed photons. This thermal energy changes electrical properties of detector material (gain, resistance, ...). Very broad wavelength response. Used at long IR and mm wavelengths. Example: bolometers. Some X-ray applications.

COHERENT RECEIVERS Respond to $E(t, \nu)$ directly, preserving <u>phase</u>. Works by interfering E from incident photons with E_{LO} from "Local oscillator." Example: radio



Amplifier Noise

Real amplifiers have sources of voltage and current noise. Equivalent circuit:



ideal amplifier with ∞ input impedance, gain g.

In general, i_A is white noise

V_A has $1/f$ component

Usually we lump the Johnson noise of the source resistor together with amplifier noises.
Since these noises have random phases they add in quadrature:

output noise power → $S_V = g^2 \left(V_A^2 + i_A^2 R_s^2 + 4 R_s k T \right)$

Consider power ratio $F \equiv \dfrac{S_V}{g^2 \, 4 R_s k T_{290}}$

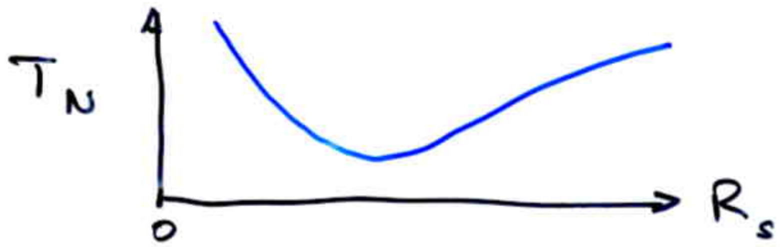
= tot. noise out at 290K / Johnson noise in R_s @ 290K

Noise figure $NF \equiv 10 \log_{10} F$

Noise temperature $\equiv T_N =$

physical temperature of R_s for which Johnson noise equals the amplifier noise.

$$T_N = \left(V_A^2 + i_A^2 R_s^2\right) / 4kR_s$$



Note T_N is a minimum when $V_A^2 = i_A^2 R_s^2$.

This is called the noise "match".

Since impedance of an audio amplifier is not matched to the source, a room temperature FET amplifier can have $T_N = 0.1\,K$!

For microwave amplifiers, the impedance is matched to avoid reflections, so

$$T_N \gtrsim T_{ambient}$$

Types of detectors

An electrical signal can be formed directly by ionization or photo-conversion. Incident radiation quanta impart sufficient energy to individual atomic electrons to form electron-ion pairs (in gases) or electron-hole pairs (in semiconductors and metals).

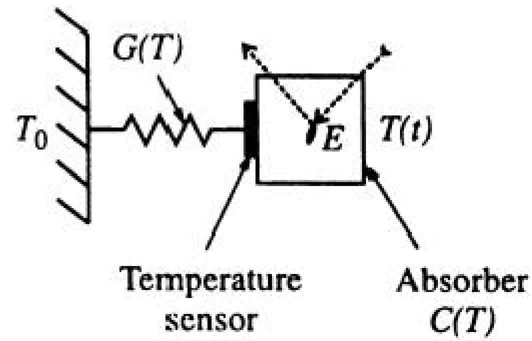
Other detection mechanisms are:

Excitation of optical states (scintillators)
Excitation of lattice vibrations (phonons)
Breakup of Cooper pairs in superconductors
Formation of superheated droplets in superfluid He

Typical excitation energies:

Ionization in gases ~30 eV
Photo-conversion in semiconductors 1 – 5 eV
Scintillation ~10 eV
Phonons meV
Breakup of Cooper Pairs meV

Other types of detectors: bolometers



Assume thermal equilibrium: If all absorbed energy *E* is converted into phonons, the temperature of the sample will increase by

$$\Delta T = \frac{E}{C}$$

where C the heat capacity of the sample (specific heat x mass).

At room temperature the specific heat of Si is 0.7 J/gK, so

$$E = 1 \text{ keV},\ m = 1 \text{ g} \quad \Rightarrow \quad \Delta T = 2 \cdot 10^{-16} \text{ K},$$

which isn't practical.

What can be done?

a) reduce mass

b) lower temperature to reduce heat capacity
"freeze out" any electron contribution, so phonon excitation dominates.

Debye model of heat capacity: $C \propto \left(\frac{T}{\Theta}\right)^3$

Example:

$m = 15\ \mu\text{g}$

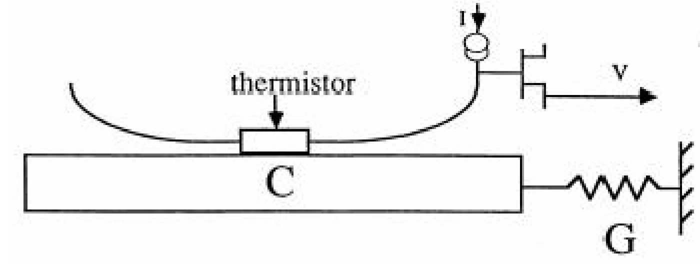
$T = 0.1$ K

Si $\Rightarrow$ $C = 4 \cdot 10^{-15}$ J/K

$E = 1$ keV $\Rightarrow$ $\Delta T = 0.04$ K

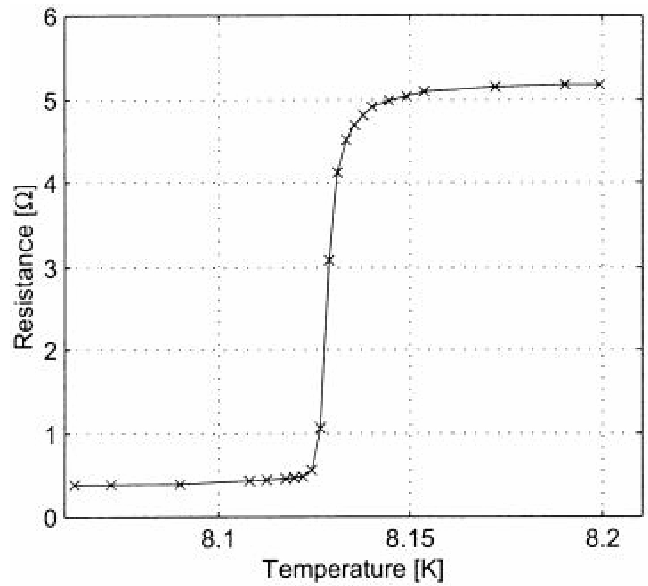
How do we measure the temperature rise?

One idea: couple thermistor to silicon and measure the resistance change:



Thermistors made of very pure semiconductors (Ge, Si) can exhibit responsivities of order 1 V/K, so a 40 mK change in temperature would yield a signal of 40 mV.

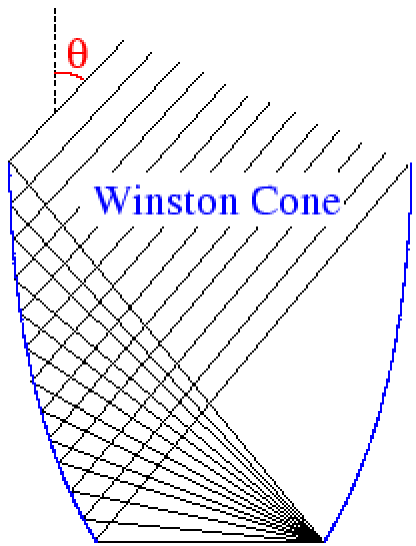
Better idea: Utilize abrupt change in resistance in transition from superconducting to normal state:



At sufficiently low temperatures the electronic contribution to the heat capacity is negligible: $C \propto \exp(-T_c/T)$

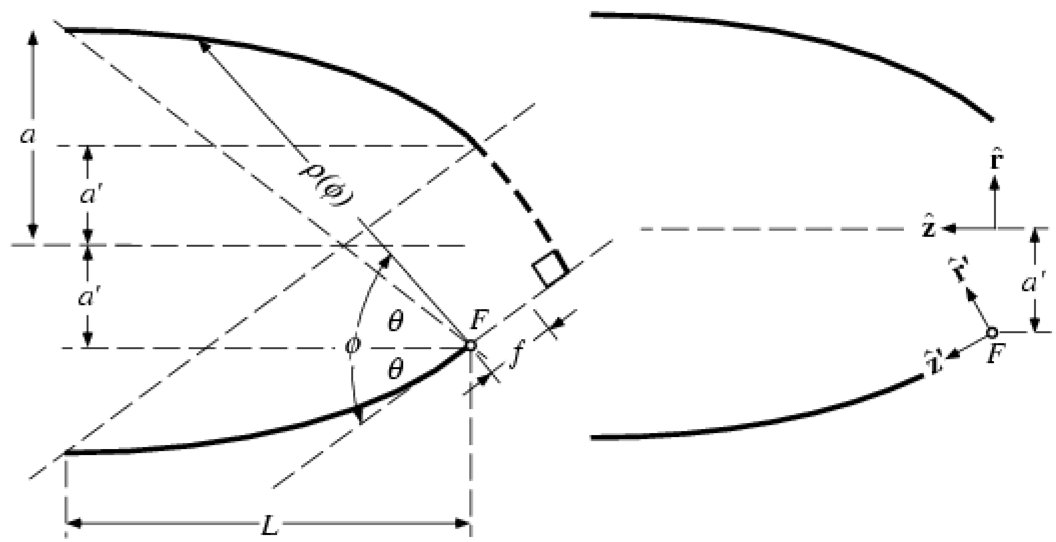
Important constraint:
Since sensor resistance of order 0.1 – 1 Ω, the total external resistance (internal resistance of voltage source and input resistance of current measuring device) must be much smaller to maintain voltage-biased operation, i.e. < 0.01 – 0.1 Ω!
Difficult to achieve at relevant frequencies.

Pseudo-optical systems



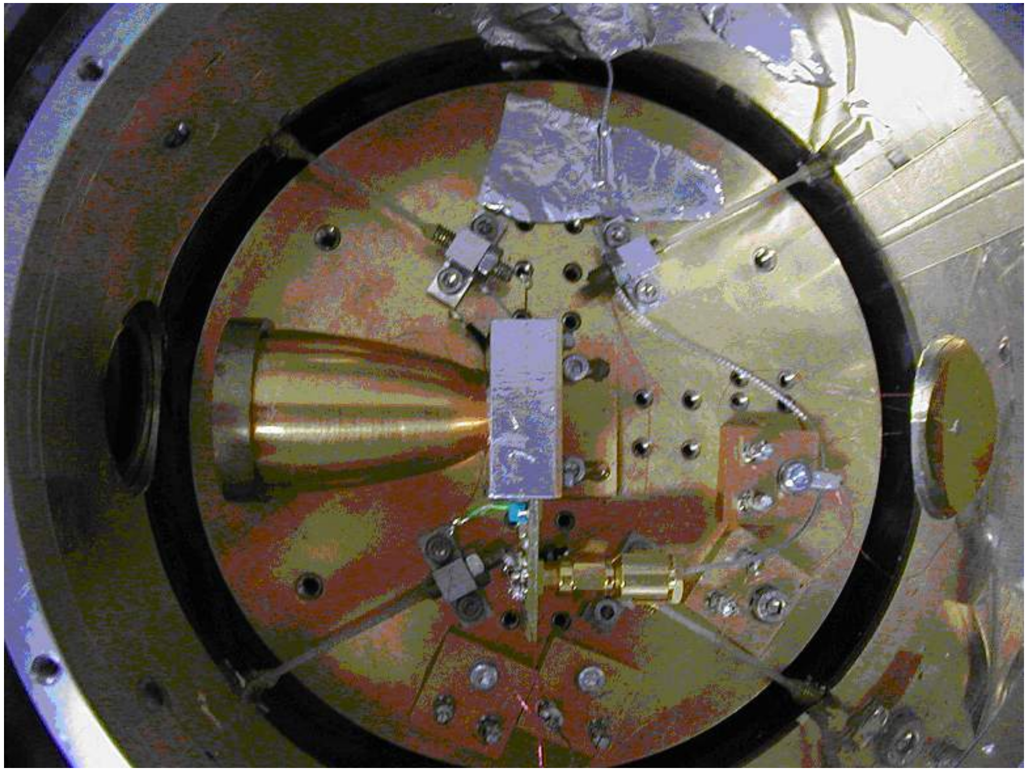
Field of view $= 2\,\Theta = 2\tan^{-1}\left(\frac{a}{f'}\right) = 2\tan^{-1}\left(\frac{a}{F'D}\right) \approx \frac{2a}{F'D},$

F-number = *F'*



Superconducting niobium bolometer system

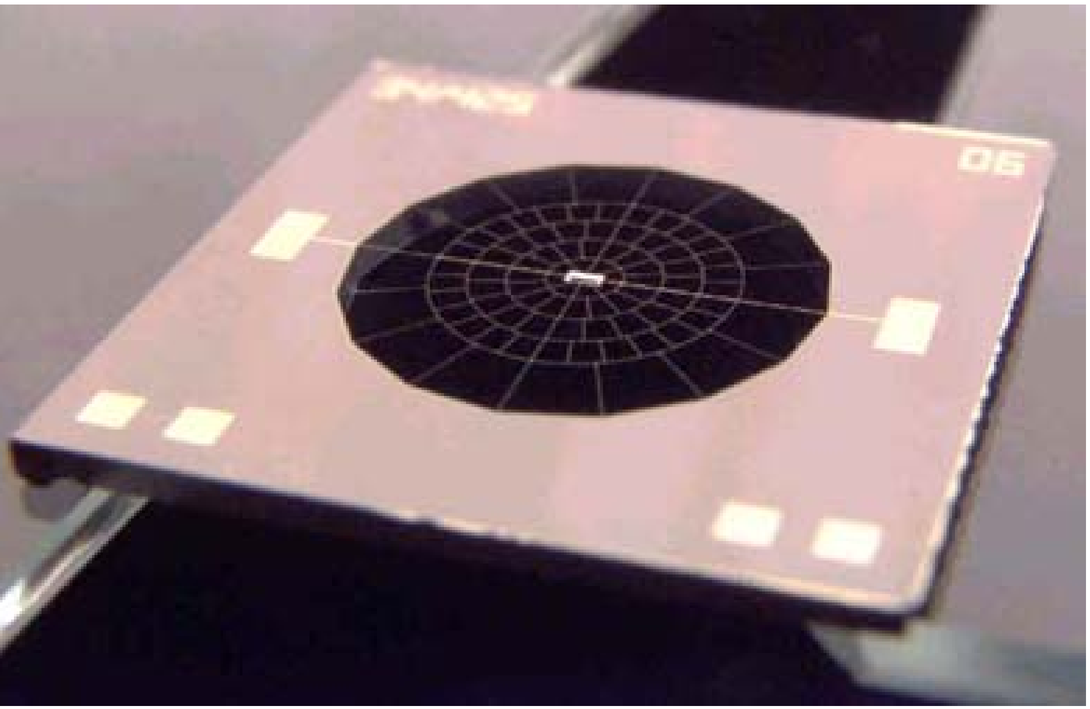
This Nb superconducting hot-electron bolometer is capable of responding to a very broad range of wavelengths. Shown here is a system with a single superconducting Nb hot electron bolometer, with Winston cone coupling optics, control thermometry and bias circuit, quasi-optical filters and a wideband noise-matched preamplifier. This detector has good sensitivity throughout the mm and IR with a one nanosecond response time.



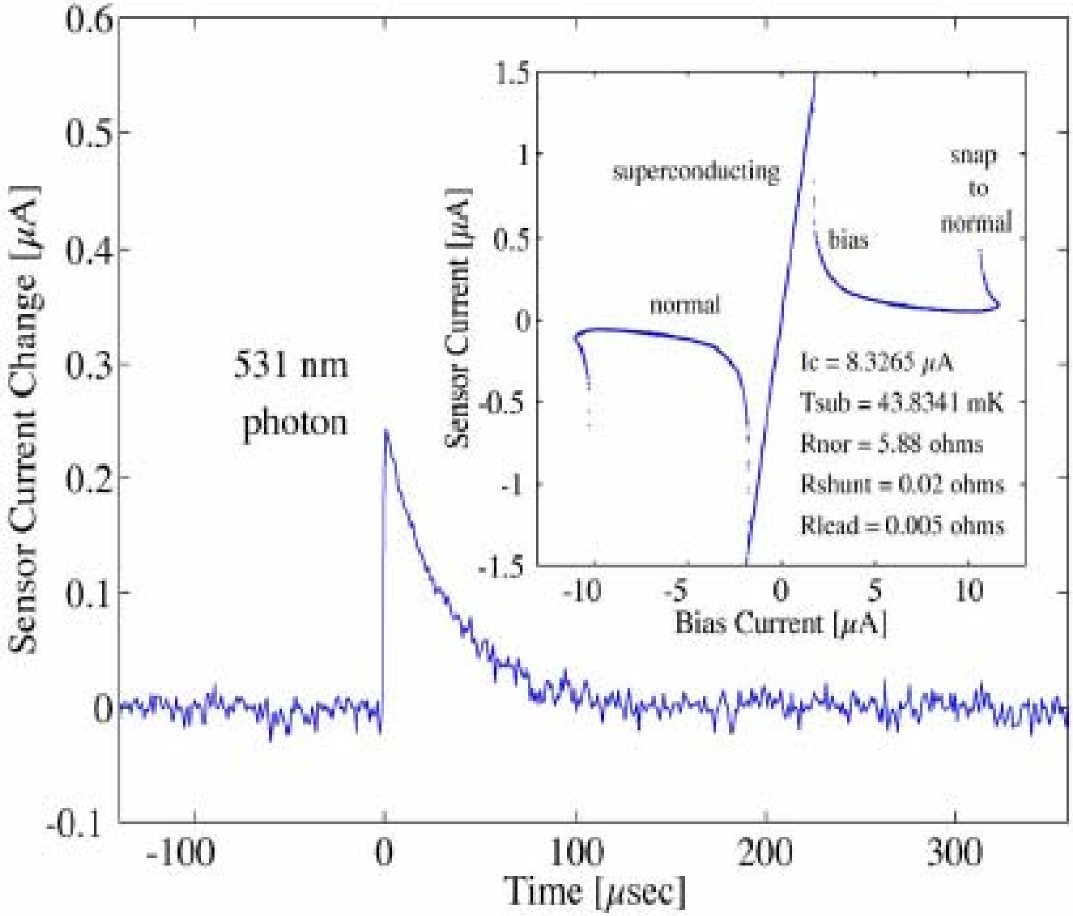
System Optical NEP: < 100 pW.Hz-1/2 measured at 300 GHz (100 kHz modulation.)
Bandwidth: > 200 MHz (τ = 1 x 10-9 second.)
Operating Temperature: 4.2K or below.
Wavelength range: > 150 microns (< 2 THz.)
Coupling Optics: 15 mm diameter at f/3.

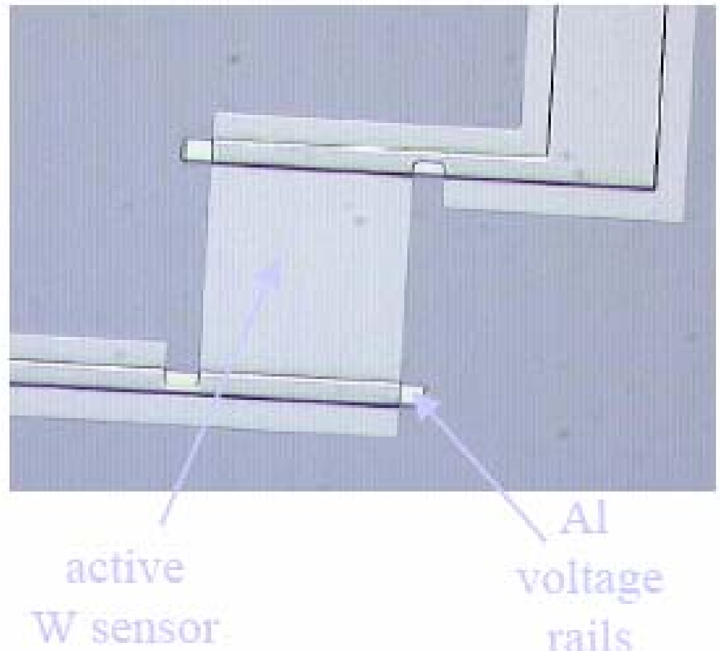
Superconducting Bolometer Array

ACBAR (Caltech)



- **Demonstration of W TES sensitivity**

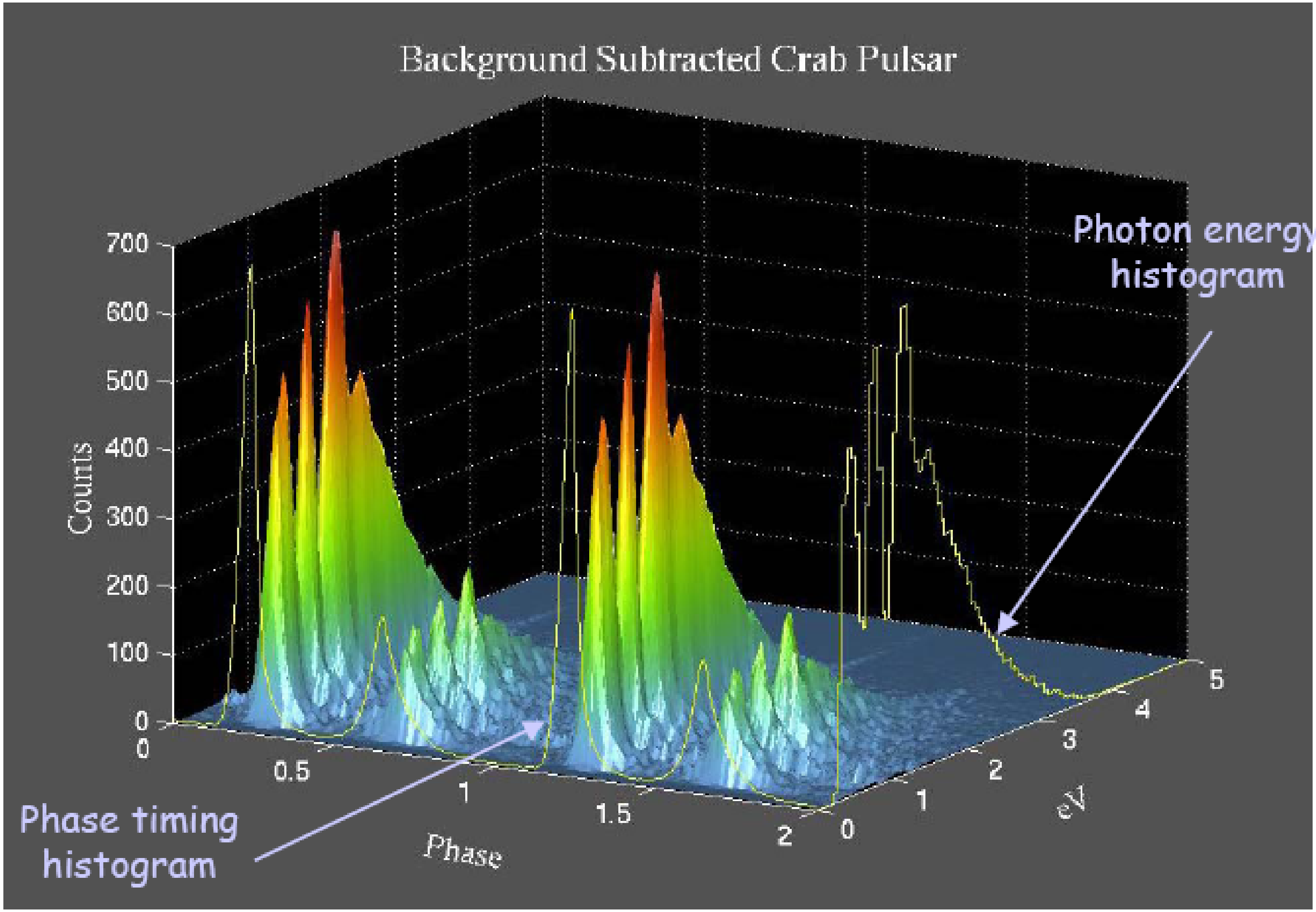




Appl. Phys. Lett. 73, 735 (1998)
B. Cabrera, R. Romani, A. J. Miller
E. Figueroa-Feliciano, S. W. Nam

Application of TES to optical astronomy

Simultaneous photon timing and spectroscopy of Crab pulsar



Cryogenic detectors

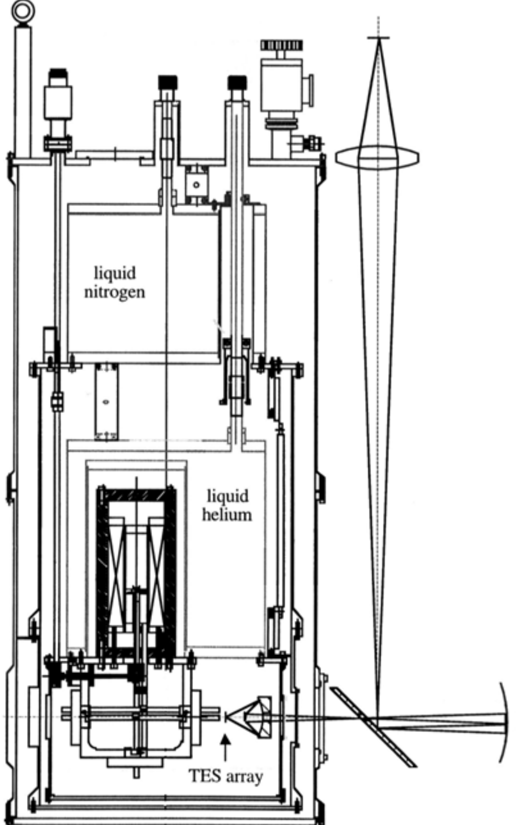
Quantum limited: photon noise in IR background is
$(NEP)^2 = 2P\,h\nu$, where P = incident power

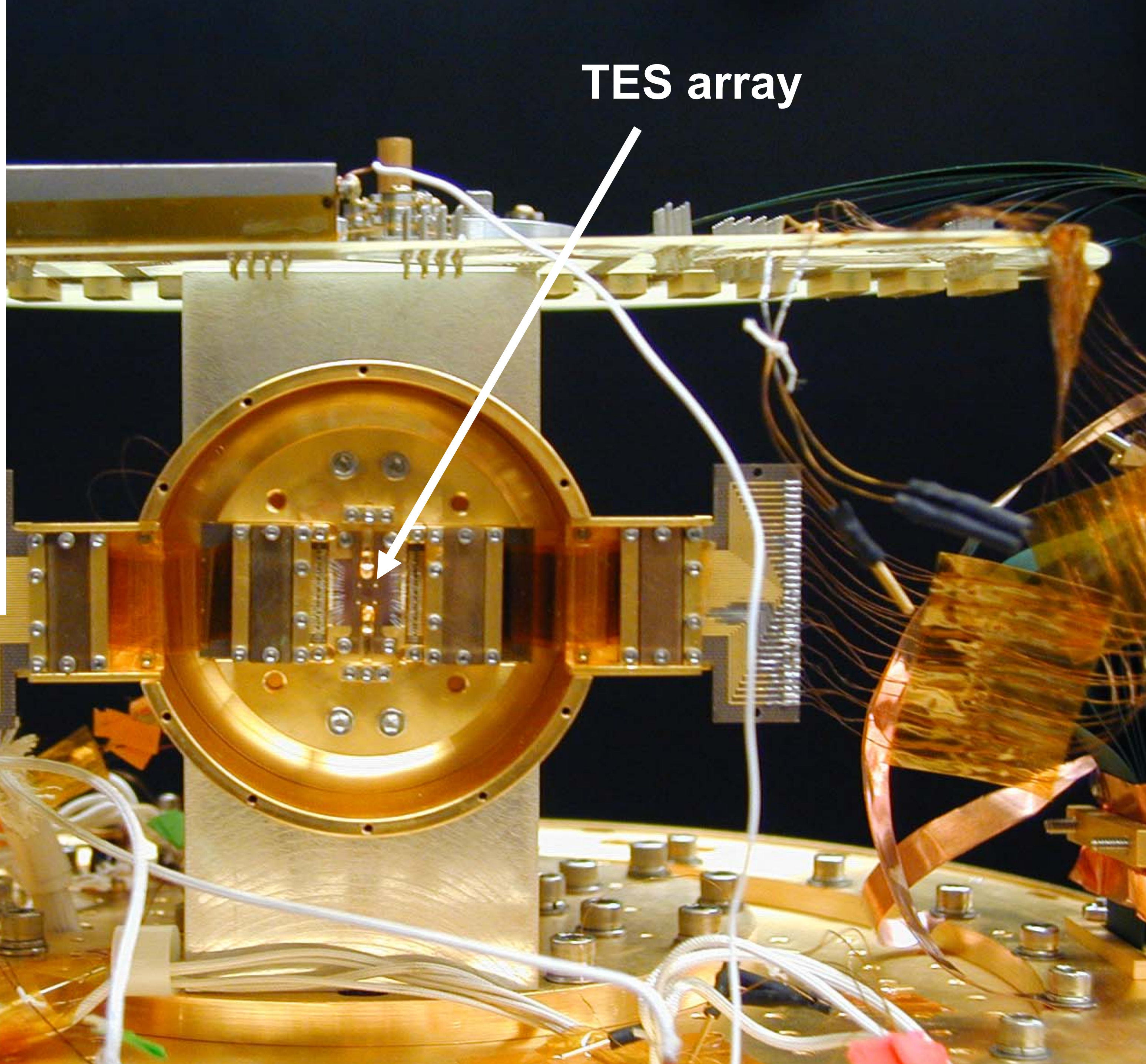
- **Sensitivity approaching quantum level at mm wavelengths**
  - **Voltage-biased superconducting transition edge sensors**

Stable operation + predictable response

- **Sensors can be fabricated using monolithic technology developed for Si integrated circuits, micro-mechanics (MEMS)**
- **Economical fabrication of large sensor arrays**
- **Open question: Readout (multiplexing of many channels)**
  Appears feasible, but much work to do

Critical for CMB Polarization SZ Cluster Search Next Generation WIMP detectors


liquid nitrogen
liquid helium
TES array


TES array