Interferometry of the intensity fluctuations in light
I. Basic theory: the correlation between photons in coherent beams of radiation

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It is shown by a quantum-mechanical treatment that the emission times of photoelectrons at different points illuminated by a plane wave of light are partially correlated, and identical results are obtained by a classical theory in which the photocathode is regarded as a square-law detector of suitable conversion efficiency. It is argued that the phenomenon exemplifies the wave rather than the particle aspect of light and that it may most easily be interpreted as a correlation between the intensity fluctuations at different points on a wavefront which arise because of interference between different frequency components of the light.

From the point of view of the corpuscular picture the interpretation is much less straightforward but it is shown that the correlation is directly related to the so-called bunching of photons which arises because light quanta are mutually indistinguishable and obey Bose–Einstein statistics. However, it is stressed that the use of the photon concept before the light energy is actually detected is highly misleading since, in an interference experiment, the electromagnetic field behaves in a manner which cannot be explained in terms of classical particles.

The quantitative predictions of the theory have been confirmed by laboratory experiments and the phenomenon has been used, in an interferometer, to measure the apparent angular diameter of Sirius: these results, together with further applications to astronomy, will be discussed in detail in later papers.

It is shown that the classical and quantum treatments give identical results when applied to find the fluctuations in the photoemission current produced by a single light beam, and the connexion between these fluctuations and the correlation between photons in coherent beams is pointed out. The results given here are in full agreement with those obtained by Kahn from an analysis based on quantum statistics: however, they differ from those derived on thermodynamical grounds by Fellgett and by Clark Jones and the reasons for this discrepancy are discussed.

1. Introduction

In this paper, the first of a series on the interferometry of intensity fluctuations in light, we shall establish theoretically the underlying principle of the technique, which is that the times of emission of photoelectrons at different points illuminated by coherent beams of light are partially correlated. The chief application of this technique is to astronomy, and it has already been successfully tested in a measurement of the angular diameter of Sirius (Hanbury Brown & Twiss 1956*b*). The existence of a correlation between photons has been denied by some authors (Brannen & Ferguson 1956) who have stated, in our view wrongly, that it is contrary to the laws of quantum mechanics. The error appears to have arisen because of a too literal reliance on the corpuscular picture of light. As Bohr has pointed out, in his

Principle of Complementarity, a particular experiment can exemplify the wave or the particle aspect of light but not both; thus the interpretation is greatly simplified, and indeed is much more likely to be correct, if one confines oneself rigidly to the use of the appropriate language and talks of photons when the energy behaves like a classical particle but otherwise talks only of waves. In the present paper, as we shall show, we are dealing essentially with an interference phenomenon which can be interpreted, on the classical wave picture, as a correlation between intensity fluctuations due to beats between waves of different frequency; the concept of a photon need only be introduced at the stage where energy is extracted from the light beam in the process of photoemission.

This does not mean that one cannot interpret the phenomenon from the corpuscular point of view: one can, but only if one is prepared to endow the photons with properties very different from those of classical particles, and in practice the corpuscular picture is more of a hindrance than a help to an interpretation of the phenomenon. Indeed if photons did behave like independent classical particles, distinguishable one from another and obeying Boltzmann statistics, the correlation between them would be identically zero. However, photons are not independent since only states symmetrical between them can occur in nature; thus they obey Bose–Einstein statistics and must be regarded as mutually indistinguishable.

The connexion between the fact that photons are bosons and the existence of a correlation between light quanta may be illustrated by the familiar example of a cavity filled with thermal radiation. In this case, as is well known, the r.m.s. fluctuations in the number of photons in an elementary cell in phase space are greater than those predicted by the classical Boltzmann statistics; as Einstein (1909) pointed out, this excess noise is essentially a wave interference effect, but it can be interpreted in the corpuscular picture as the so-called 'bunching' of photons (Clark Jones 1953).

In principle this 'bunching' of photons could be measured directly if a single photocathode were illumined by a coherent beam of light, since the fluctuations in the photoemission current should be slightly greater than the pure noise fluctuations which would arise if the photoelectrons were emitted completely independently. In practice the difference between photon and shot noise, which we have called the excess photon noise, is too small to be detected conveniently with one photocathode (Fürth & MacDonald 1947), being swamped by effects such as space-charge smoothing in the photocell or fluctuations in the multiplication process in a photomultiplier. However, the 'bunching' can be measured with two separate phototubes, the cathodes of which lie in the same cell in phase space or, in other words, are illumined by coherent beams of light.† In this arrangement the shot noise currents, the space-charge smoothing effects and the multiplication noise in the two phototubes are uncorrelated, and thus the small correlation between the fluctuations in the two currents can be detected if the observations are carried out over a sufficiently long time. This correlation can only arise if there is a corresponding correlation in the

† The connexion between the extent, in real space, of an elementary cell in phase space and the volume over which a light beam may be regarded as coherent is not perhaps self-evident and it is therefore examined in appendix I.

time of emission of photoelectrons from the two cathodes, and it follows that this latter phenomenon is related to the fact that photons obey Bose–Einstein statistics. It is of course possible, by means of quantum statistics, to develop the theory given in this paper entirely in terms of the particle picture, as has been done by Kahn (1957); however, we have chosen an alternative approach which emphasizes that the correlation between photons is essentially an interference effect related to the wave picture rather than to the corpuscular aspect of light.

Experiments to measure directly the correlation in the arrival times of photons with coincidence counters have been carried out by Ádám, Jánossy & Varga (1955) and, with more sensitive equipment, by Brannen & Ferguson (1956), but with a negative result. However, as we have pointed out elsewhere (Hanbury Brown & Twiss 1956*c*), under the conditions of these experiments the expected correlation would have been far too small to be detected. We have carried out independently (Hanbury Brown & Twiss 1956*a*) a similar experiment in which we measured the correlation between fluctuations in the emission currents† of two phototubes, under conditions where the expected signal to noise ratio was of the order 10 to 1, and we have obtained a positive result in satisfactory quantitative agreement with theory. However, the detailed interpretation of this experiment will be left to a later paper of this series since the analysis is complicated by the fact that the light beam was not fully coherent over the surfaces of the photocathodes. In the present paper we shall consider only the idealized case of a plane wave of linearly polarized light in order to present the basic theory in the simplest form.

The phenomenon we are discussing is a general characteristic of an electromagnetic radiation field and will therefore occur not only at optical but also at radio wavelengths. The existence of the effect in the latter case has been demonstrated, implicitly, by experiments with an 'intensity' interferometer which has been used to measure the angular diameter of discrete radio sources (Hanbury Brown, Jennison & Das Gupta 1952). In these experiments energy was extracted from the electromagnetic field by two separate aerials, corresponding to the apertures of the phototubes in the optical experiment, and was then rectified by two square law detectors which correspond to the two photoelectric cathodes. The correlation between the fluctuations in the output currents of the two detectors was measured and was found to be equal to the theoretical value as calculated by classical electromagnetic theory.

The general theory of this radio interferometer has been given elsewhere (Hanbury Brown & Twiss 1954), but in the rather complex form required for practical applications to radio-astronomy. To bring out the connexion between the radio and the optical case we shall first develop a simple classical theory for the correlation between the intensity fluctuations at different points in space for the idealized case where the incident radiation field is a plane wave of radio frequency.

† The correlation was measured in this way, and not with a coincidence counter as in the experiments of Ádám *et al.* (1955), because the latter technique is not practical for the measurements on stars to which our work was primarily directed.

2. The classical theory of the intensity fluctuations in a plane electromagnetic wave

(a) *The intensity fluctuations in a plane wave*

Let us assume that the frequency components of the incident electromagnetic plane wave are confined to a limited region of the radio-frequency spectrum defined by

$$\nu_1 < \nu < \nu_2,$$

such that

$$\nu_1 > \nu_2 - \nu_1.$$

If the voltage induced by this radiation field in an aerial of aperture A is rectified in a square-law detector, then the low-frequency fluctuations in the output current of the detector can be expressed as a sum of the beats between the different radio-frequency components of the electromagnetic wave and correspond to the intensity fluctuations in the incident radiation. It is obvious that the amplitude and phase of these low-frequency fluctuations in the detector output current are the same at any point on the wavefront of a plane wave: so if signals are picked up by two separate aerials and rectified in separate square-law detectors, the low-frequency fluctuations in the two output currents will be perfectly correlated so long as the effects of shot noise in the detector current can be neglected. The fact that this correlation is equally to be expected, on a classical theory, at optical wavelengths appears to have been overlooked.

To develop this argument in a quantitative form, which will later be compared with the results obtained by a quantum theory, we proceed as follows. By a suitable choice of gauge a linearly polarized wave of electromagnetic radiation can be completely described by a vector potential $\mathfrak{A}$ with a single component perpendicular to the direction of propagation. If the observation is of duration T, this component can be represented by a Fourier series,

$$\mathfrak{A} = \sum_{r=1}^{\infty} q_r \exp\left[\frac{2\pi i r}{T}(t+\mathbf{k}.\mathbf{x})\right] + q_r^* \exp\left[-\frac{2\pi i r}{T}(t+\mathbf{k}.\mathbf{x})\right], \tag{2·1}$$

where q_r, q_r^* are quantities determining the amplitude and phase of the rth Fourier component, and the sign of $\mathbf{k}.\mathbf{x}$ is that appropriate to an inward travelling wave. In the present case we are assuming that q_r is zero except when $\nu_1 T < r < \nu_2 T$.

In a classical theory q_r is a complex number such that

$$q_r q_r^* = \left(\frac{p_r}{8\pi^2\nu_r^2 T}\sqrt{\frac{\mu_0}{\epsilon_0}}\right) \tag{2·2}$$

where p_r/T is the power flow across unit area perpendicular to the direction of propagation associated with the rth Fourier component of frequency ν_r, where

$$\nu_r = r/T, \tag{2·3}$$

and where $(\mu_0/\epsilon_0)^{\frac{1}{2}}$ is the characteristic impedance of free space. If we define a quantity n_r by the equation

$$n_r h\nu_r = p_r, \tag{2·4}$$

then n_r/T may formally be identified with the average number of quanta of energy $h\nu_r$ crossing unit area in unit time, and we may put

$$q_r = \left(\frac{h}{8\pi^2\nu_r}\sqrt{\frac{\mu_0}{\epsilon_0}}\right)^{\frac{1}{2}} n_r^{\frac{1}{2}} \exp i\phi_r, \tag{2·5}$$

where ϕ_r is the phase of the rth Fourier component of the vector potential at the wavefront defined by

$$t + \mathbf{k}.\mathbf{x} = 0. \tag{2·6}$$

In the limiting case as $T \to \infty$, n_r is the average number of quanta per unit frequency bandwidth.

In what follows we shall assume that the phases of the different Fourier components are quite uncorrelated so that we may take the values of ϕ_r to be a set of independent random variables distributed with uniform probability over the range $0 < \phi_r < 2\pi$. This assumption is certainly valid as long as the radiation can be described by a stationary time series. Even when this is not the case, as when the electromagnetic energy is produced in bursts, the phases of the Fourier components of the radiation received by the observer will be effectively uncorrelated as long as the region of the source over which the intensity fluctuation is coherent is sufficiently small.

The voltage $V(t)$ produced across the input terminals to the square law detector by the vector potential defined by (2·1) will be of the form

$$V(t) = \sum_{r=1}^{\infty} \beta_r q_r \exp\left[\frac{2\pi i r}{T}(t + \mathbf{k}.\mathbf{x}_1)\right] + \beta_r^* q_r^* \exp\left[-\frac{2\pi i r}{T}(t + \mathbf{k}.\mathbf{x}_1)\right], \tag{2·7}$$

where $\mathbf{x}_1$ are the co-ordinates of the phase reference point and β_r is a complex quantity such that $\beta_r \beta_r^*$ is linearly proportional to the aerial aperture A and to the aerial efficiency at frequency r/T.

If we substitute from (2·5) and (2·7) in the equation

$$\mathrm{i}(t) = bV^2(t), \tag{2·8}$$

then the low-frequency components in the output current of the square-law detector are given by an expression of the general form

$$i(t) = eA \sum_{r=1}^{\infty} \frac{\alpha_r n_r}{T} + 2eA \sum_{r>s}^{\infty} \sum_{s=1}^{\infty} \left(\frac{\alpha_r \alpha_s n_r n_s}{T^2}\right)^{\frac{1}{2}} \cos\left[\frac{2\pi}{T}(r-s)(t + \mathbf{k}.\mathbf{x}_1) + \phi_r - \phi_s\right], \tag{2·9}$$

where e is the electronic charge and α_r is defined by the equation

$$\alpha_r = \frac{\beta_r \beta_r^*}{eA} \frac{h}{4\pi^2 \nu_r} \sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)} b. \tag{2·10}$$

This unconventional and somewhat clumsy symbolism has been adopted so that a direct comparison may be made with the quantum treatment of the optical case in which the symbol α_r will correspond to the photocathode quantum efficiency.

Let us now suppose that the a.c. fluctuations in the detector output are passed through a filter, with a frequency response $F(f)$ which does not transmit d.c. so that

$$F(0) = 0, \tag{2·11}$$

then $J(t)$ the output current of this filter may be written

$$J(t) = A \sum_{r>s}^{\infty} \sum_{s=1}^{\infty} \left(\frac{\alpha_r \alpha_s n_r n_s}{T^2}\right)^{\frac{1}{2}} \left\{ F\left(\frac{r-s}{T}\right) \exp \mathrm{i} \left[\frac{2\pi(r-s)}{T}(t+\mathbf{k}.\mathbf{x}_1)+(\phi_r-\phi_s)\right] + F^*\left(\frac{r-s}{T}\right) \exp - \mathrm{i} \left[\frac{2\pi(r-s)}{T}(t+\mathbf{k}.\mathbf{x}_1)+(\phi_r-\phi_s)\right] \right\}. \quad (2{\cdot}12)$$

This expression may be simplified if the filter bandwidth is so narrow that $\alpha_r n_r \simeq \alpha_s n_s$ for all values of r and s for which the frequency $(r-s)/T$ lies in the filter passband. In this case if we introduce two new indices l, m defined by

$$\tfrac{1}{2}(r+s) = l, \quad r-s = m, \quad (2{\cdot}13)$$

we have that

$$J(t) = A \sum_{m=1}^{M} \sum_{l=T\nu_1}^{T\nu_2} \frac{a_l n_l}{T} \left\{ F\left(\frac{m}{T}\right) \exp \mathrm{i} \left[\frac{2\pi m}{T}(t+\mathbf{k}.\mathbf{x}_1)+(\phi_r-\phi_s)\right] + F^*\left(\frac{m}{T}\right) \exp - \mathrm{i} \left[\frac{2\pi m}{T}(t+\mathbf{k}.\mathbf{x}_1)+(\phi_r-\phi_s)\right] \right\}, \quad (2{\cdot}14)$$

where M/T, the highest beat frequency passed by the filter, is very much less than $\nu_2-\nu_1$, the bandwidth of the incident radiation, and where ϕ_r, ϕ_s are independent random variables distributed with uniform probability over the range

$$0 < \phi_r < 2\pi; \quad 0 < \phi_s < 2\pi.$$

This result will now be used to derive expressions for the correlation between intensity fluctuations at different points on the wavefront and for the mean square value of the intensity fluctuations at a single point.

(*b*) *The correlation between intensity fluctuations at different points on the wavefront*

Let us consider the case where the plane electromagnetic wave is incident on two aerials with apertures A_1, A_2 and phase reference points $\mathbf{x}_1$, $\mathbf{x}_2$ respectively. If $J_1(t)$, $J_2(t)$ are the a.c. output currents of the two low frequency filters, which we shall assume to have identical characteristics, the correlation $C(T_0)$ between these two currents, averaged over a time interval T_0, is given by

$$C(T_0) = \frac{1}{T_0}\int_0^{T_0} J_1(t-t_0)\, J_2(t)\, \mathrm{d}t, \quad (2{\cdot}15)$$

where

$$t_0 = \mathbf{k}.(\mathbf{x}_1-\mathbf{x}_2) \quad (2{\cdot}16)$$

is the difference in time between the arrival of the incident radiation at the two aerials, and where T_0 may have any value less than T.

For our present purposes the quantity of interest is $\bar{C}$, the ensemble average of $C(T_0)$ taken over an infinite number of independent time intervals each of length T_0, which is equal to the time average

$$\lim_{T_0 \to \infty} C(T_0)$$

in the present case where the fluctuations are determined by a stationary time series.

For the classical radio case this calculation is very straightforward, since it is only necessary to average over the random radio-frequency phases. Terms in $C(T_0)$ which depend on ϕ_r, ϕ_s will average to zero and so we have immediately that

$$\bar{C} = 2e^2 A_1 A_2 \sum_{l=\nu_1 T}^{\nu_2 T} \frac{\alpha_l^2 n_l^2}{T} \sum_{m=1}^{M} \frac{|F(m/T)|^2}{T}, \tag{2·17}$$

which, as one would expect, is independent of T_0.

In the limiting case $T \to \infty$, so we may replace the sums in (2·17) by integrals on putting $1/T = \mathrm{d}\nu$ when we have that

$$\bar{C} = 2e^2 \int_{\nu_1}^{\nu_2} A_1 A_2 \alpha^2(\nu)\, n^2(\nu)\, \mathrm{d}\nu \int_0^\infty F^2(f)\, \mathrm{d}f. \tag{2·18}$$

(c) *The mean square value of the intensity fluctuations*

If, for the moment, we ignore the effects of shot noise in the current of the square-law detector, the mean square fluctuations $\overline{j_C^2}$ in the output current of the filter may be defined as the ensemble average of

$$\frac{1}{T_0} \int_0^{T_0} J^2(t)\, \mathrm{d}t,$$

where $J(t)$ is given by (2·14). Accordingly, $\overline{j_C^2}$ is given by (2·18) with $A_1 = A_2 = A$.

If a current I_0 flows in the detector circuit the mean square fluctuations are increased by the shot noise term $\overline{j_N^2}$ which, in the absence of space-charge smoothing (Rice 1944), is given by

$$\overline{j_N^2} = 2eI_0 \int_0^\infty |F(f)|^2\, \mathrm{d}f. \tag{2·19}$$

Now from (2·9) the incident radiation field increases the average current I_0 in the square-law detector by J_0, where

$$J_0 = eA \sum_{r=1}^{\infty} \alpha_r n_r / T, \tag{2·20}$$

so, in the limiting case, as $T \to \infty$,

$$J_0 = e \int_0^\infty A\alpha(\nu)\, n(\nu)\, \mathrm{d}\nu. \tag{2·21}$$

The total mean square fluctuations $\overline{J^2}(t)$ in the filtered output current of the square-law detector due to the incident radiation field are therefore given by

$$\overline{J^2(t)} = \overline{j_N^2} + \overline{j_C^2} = 2e^2 \left[\int_0^\infty A\alpha(\nu)\, n(\nu)\, \mathrm{d}\nu + \int_0^\infty A^2\alpha^2(\nu)\, n^2(\nu)\, \mathrm{d}\nu \right] \int_0^\infty |F(f)|^2\, \mathrm{d}f, \tag{2·22}$$

since the noise currents j_N and j_C are uncorrelated.

The first term in (2·22) represents the *shot-noise* term due to the discrete nature of the particles carrying the detector current, while the second term, which is due to beats between the different Fourier components of the incident radiation field, may be called the *wave interaction* noise.

In a typical radio case $An(\nu)$, which is effectively equal to the number of quanta extracted from the radiation field by the aerial in unit time and unit bandwidth, is of the order of 10^5, while $\alpha(\nu)$, which is effectively equal to the average number of electrons transported from cathode to anode of the square-law detector by the incidence of a single photon, might be of the order of 10^6. Under these conditions $\overline{j_C^2}$ exceeds $\overline{j_N^2}$ by a factor of 10^{11}, so that the contribution of the latter is completely negligible.

However, at optical wavelengths $An(\nu)$ is of the order of 10^{-4} in a typical case, while $\alpha(\nu)$, the quantum efficiency of the photocathode, is of the order of 10^{-1}. Under these conditions everything is reversed and the classical theory would lead one to expect that $\overline{j_C^2}$ would be smaller than $\overline{j_N^2}$ by a factor $\sim 10^{-5}$. Admittedly it is not obvious that the quantitative predictions of a classical and determinist wave theory will be valid for the optical case, but it is shown below that indeed they are and that the wave interaction noise is simply another name for the excess photon noise, due on the corpuscular picture to the fact that photons obey Bose–Einstein statistics.

3. The fluctuations in the photoelectric emission due to a plane wave of light

(a) *The probability of photoelectric emission by a plane wave of light*

In order to calculate the correlation between the times of emission of photoelectrons at different points of a wavefront and to find the mean square fluctuations in the photoemission current from a given photocathode, we shall first obtain an expression for the probability of photoemission in terms of the observables of the incident beam of light.

In a quantum theory one must regard the quantities q_r, q_r^*, which occur in (2·1), as operators rather than as numbers, and the quantities n_r, $\exp \mathrm{i}\phi_r$, which correspond to the action and angle variables of the equivalent harmonic oscillator, are also operators satisfying commutation relations of the form (Heitler 1954),

$$n_r \exp(\mathrm{i}\phi_s) - \exp(\mathrm{i}\phi_s)\, n_r = \delta_{rs} \exp(\mathrm{i}\phi_s), \tag{3·1}$$

$$n_r \exp(-\mathrm{i}\phi_s) - \exp(-\mathrm{i}\phi_s)\, n_r = -\delta_{rs} \exp(-\mathrm{i}\phi_s), \tag{3·2}$$

where δ_{rs} is the familiar Kronecker symbol and (3·2) is the complex conjugate of (3·1).

In the standard treatment of the interaction between the matter and radiation fields, as given by Dirac (1947), one calculates the probability of a transition in which a photon is absorbed from a specific Fourier component of the radiation field so that the number of quanta associated with this component changes by unity. However, this procedure can clearly not be used to analyse an experiment in which we measure the correlation between the times of arrival of photons at different points of a wavefront, since, if the time of arrival of a photon is known to an accuracy Δt, the uncertainty, $\Delta E \equiv h\Delta\nu$, in the energy must satisfy the inequality

$$\Delta E \Delta t = h, \tag{3·3}$$

or

$$\Delta\nu \Delta t = 1. \tag{3·4}$$

If the particular Fourier component with which a specific photon is to be associated is known then $\Delta\nu = 1/T$, where T is the total observing time, and one has no knowledge whatever as to the actual moment, in the observation period, when the photon arrived.

It follows that the action and angle variables of the radiation field are not observables for the conditions under which one would look for a correlation between the arrival times of photons. As we have just shown in the classical analysis of the radio-frequency case the intensity fluctuations depend upon the beat frequencies between the different radio-frequency components of the incident radiation rather than upon the radio-frequency components themselves, while the correlation between the intensity fluctuations is determined by the amplitudes and relative phases of these beat frequencies.

When interpreting interference phenomena according to the corpuscular theory of radiation, it has been emphasized by Dirac (1947) that one must not talk of interference between two different photons, which never occurs, but rather of the interference of a photon with itself. This point was originally made for the case of spatial interference, as in an interferometer, but the arguments on which it is based are equally valid for temporal interference as in the phenomenon of a beat frequency. Accordingly, in the corpuscular theory, one must not interpret a beat frequency as an interference between photons of different energy, but rather as a phenomenon caused by the uncertainty in the energies of the individual photons which may be associated with either of the two Fourier components of the radiation field, the interference of which gives the beat frequency.

It follows that the observables appropriate to the measurement of a beat frequency are the *relative* phases of the two Fourier components and the *total* number of quanta associated with the two components. As is well known (Heitler 1954), these quantities can be measured simultaneously without violating the uncertainty principle since they are characterized by operators of the form

$$n_r + n_s \quad \text{and} \quad \exp \mathrm{i}(\phi_r - \phi_s)$$

which commute. To prove this we have from (3·1) and (3·2) that

$$\begin{aligned}(n_r+n_s)\exp \mathrm{i}(\phi_r-\phi_s) &= \exp \mathrm{i}(\phi_r-\phi_s)n_r + \exp \mathrm{i}(\phi_r-\phi_s) \\ &\qquad + \exp \mathrm{i}(\phi_r-\phi_s)n_s - \exp \mathrm{i}(\phi_r-\phi_s) \\ &= \exp \mathrm{i}(\phi_r-\phi_s)\,(n_r+n_s). \end{aligned} \tag{3·5}$$

For the specialized purposes of this paper, in which one is concerned simply with the fluctuations in the cathode currents of photocells, one may therefore discuss the interaction of the radiation field and the photocathode by a simplified theory in which the radiation field is characterized by a set of commuting operators and may therefore be treated classically. This procedure takes no specific account of the fact that the emission of a photoelectron reduces the total number of photons in the radiation field by one, but then there is no *a priori* knowledge of this number, still less of the actual distribution of these photons, with energy: all that is known, from a study of the light source, is the *average* number of photons arriving in unit time in unit bandwidth together with the fact that the fluctuations in the number of incident photons are controlled by Bose–Einstein statistics. It is this indeterminacy, basic to the existence of a correlation between photons, which makes it possible to use a classical treatment for the radiation and impossible to use the standard quantized field treatment of the photoelectric effect: the latter applies rigorously to an experiment where the energy of the incident photon and the momentum of the emitted electron can both be known to the maximum accuracy permitted by the uncertainty principle.

In what follows we shall assume that Ψ represents the total wave function for the electrons and ions forming the photocathode when acted upon by the incident radiation field. Then Ψ will be a solution of the Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial \Psi}{\partial t} = (H_0 + H_1)\Psi, \tag{3·6}$$

where H_0 is the Hamiltonian for the matter field alone, and H_1, the interaction energy, is of the form

$$H_1 = \sum_l \rho_l \mathbf{v}_l \mathfrak{A}(\mathbf{x}_l), \tag{3·7}$$

where $\mathbf{v}_l$ is a dynamical variable describing the lth particle of charge ρ_l at the point $\mathbf{x}_l$, and $\mathfrak{A}(\mathbf{x}_l)$ is the vector potential acting on the lth particle.

From (2·1) and (2·5) the expression for H_1 may be written

$$H_1 = \sum_l \rho_l \mathbf{v}_l \sum_{r=1}^{\infty} \left(\frac{\hbar}{4\pi\nu_r T}\sqrt{\frac{\mu_0}{\epsilon_0}}\right)^{\frac{1}{2}} \{n_r^{\frac{1}{2}} \exp(+\mathrm{i}\phi_r) \exp(2\pi \mathrm{i}\nu_r(t+\mathbf{k}\,.\,\mathbf{x}_l))$$
$$+ \exp(-\mathrm{i}\phi_r)\, n_r^{\frac{1}{2}} \exp(-2\pi\mathrm{i}\nu_r(t+\mathbf{k}\,.\,\mathbf{x}_l))\} \tag{3·8}$$

If the wave function Ψ_0 satisfying the zero-order equation

$$\mathrm{i}\hbar \frac{\partial \Psi_0}{\partial t} = H_0 \Psi_0 \tag{3·9}$$

can be found, one can use a perturbation procedure to determine the first-order approximation to the exact solution from the equation

$$\mathrm{i}\hbar \frac{\partial \Psi_1}{\partial t} - H_0 \Psi_1 = H_1 \Psi_0. \tag{3·10}$$

For our present purposes there is no need to derive a detailed theory for the photoelectric effect since quantitative data, such as the dependence of quantum efficiency on frequency, or the lower limit to the time delay of photoemission, can be taken from experiment. The important thing to note is that the interaction energy is linearly dependent upon the vector potential of the radiation field as, therefore, is the first-order perturbation in the wave function of the matter field. However, this is no longer the case in the second- and higher-order perturbation terms which describe processes in which several photons are simultaneously emitted or absorbed. Such processes are of two kinds. In the first, several photons are involved in the emission of a single photoelectron, but such events are very rare and can be ignored without significant error. In the second, two or more electrons are emitted in a process, in which each photoemission absorbs a single photon, which is clearly related to the problem of the coherent emission of photoelectrons. In a fully rigorous treatment one would have to use a higher-order perturbation theory to analyze this case but we shall make the simplifying assumption that the combined probability of obtaining two photoemissions in a very small time interval from areas A_1 A_2 of a photocathode is equal to the product of the probability of obtaining a photoemission from each area separately. Clearly this assumption is valid in the physically important case when the areas A_1 A_2 belong to quite separate photocathodes illumined by coherent light beams since the actual processes of photoemission in the two photocathodes are quite independent. The assumption will also be valid for a single photocathode as long as the fractional volume over which appreciable electronic interaction can take place inside the photocathode is very small compared with unity, a condition that will always be met in practice.

The solution for Ψ_1, corresponding to the *absorption* of a photon by an electron which is then emitted from the photocathode, is of the general form

$$\Psi_1 = \sum_l \sum_{r=1}^{\infty} \eta_{lr} \exp(-\mathrm{i}\phi_r) n_r^{\frac{1}{2}} \exp\{-2\pi \mathrm{i}\nu_r(t+\mathbf{k}.\mathbf{x}_l)\}, \tag{3·11}$$

where η_{lr} is a complex quantity involving the lth particle of the matter field and the rth component of the radiation field. The terms in H_1 proportional to $\exp\{2\pi\mathrm{i}\nu_r(t+\mathbf{k}.\mathbf{x}_l)\}$ do not contribute since they correspond to processes involving the *emission* of a photon by the particles of the photocathode.

The probability of a single photoemission in time $\mathrm{d}t$ is then proportional to

$$\mathrm{d}t\int \Psi_1^* \Psi_1 \,\mathrm{d}\tau,$$

where the integral is taken over the volume of the photocathode and a summation is made over all the particles of the matter field. If we assume that the photocathode of area A_1 is placed normal to the incident plane light wave so that $(\mathbf{x}_1)$ are the co-ordinates of the midpoint of the cathode, then

$$P(\mathbf{x}_1, t) \equiv \int \Psi_1^* \Psi_1 \,\mathrm{d}\tau \tag{3·12}$$

is given by an expression of the form

$$P(\mathbf{x}_1, t) = \sum_{r=1}^{\infty} A_1 \frac{\alpha_r n_r}{T} + 2A_1 \sum_{r>s}^{\infty} \sum_{s=1}^{\infty} \left(\frac{\alpha_r \alpha_s n_r n_s}{T^2}\right)^{\frac{1}{2}}$$
$$\times \cos(2\pi(\nu_r - \nu_s)(t+\mathbf{k}.\mathbf{x}_1) + (\phi_r - \phi_s) - (\theta_r - \theta_s)), \tag{3·13}$$

where θ_r/ν_r, θ_s/ν_s determine the delays in the emission of a photoelectron after the absorption of photons of energy $h\nu_r$, $h\nu_s$ respectively. As the delay in photoemission is known experimentally (Forrester, Gudmundsen & Johnson 1955) to be much less than 10^{-10} s while, because of the limited amplifier bandwidths and the spread in transit time through the photomultiplier tube, the beat frequencies which are significant in a practical case all lie below 10^8 or 10^9 c/s, we may put

$$\theta_r - \theta_s = 0$$

in (3·13) without introducing significant error.

The quantity α_r is simply the cathode quantum efficiency for a normally incident plane wave of frequency r/T and, as in the analysis for the radio case, we shall assume that α_r is a smoothly changing function of frequency effectively constant over the maximum beat frequency bandwidth that can arise in practice. It would be difficult to establish this assumption experimentally since the cathode quantum efficiency is normally measured with a light beam of bandwidth large compared to 10^8 c/s and any rapid changes in α_r with frequency would be smoothed out, but it is almost certainly valid in view of the appreciable energy spread of the electrons inside the photocathode. We shall also assume that the quantity n_r, which represents the average number of quanta of energy crossing unit area in unit time, is a smoothly varying function of frequency effectively constant over the beat frequency bandwidth. As before we now introduce two indices l, m defined by

$$\tfrac{1}{2}(r+s) = l, \quad r-s = m,$$

when $P(\mathbf{x}_1, t)$ is given by

$$P(\mathbf{x}_1, t) = \sum_{r=1}^{\infty} A_1 \frac{\alpha_r n_r}{T} + 2A_1 \sum_{m=1}^{M} \sum_{l=L_1}^{L_2} \frac{\alpha_l n_l}{T} \cos\left(\frac{2\pi m}{T}(t+\mathbf{k}.\mathbf{x}_1) + (\phi_r - \phi_s)\right), \qquad (3\cdot14)$$

a result that can be compared with (2·9).

(*b*) *The correlation between fluctuations in the emission currents of two separate photomultipliers*

Let us consider the case where a linearly polarized plane wave of light is normally incident on two separate photocathodes of areas A_1, A_2 centred at $\mathbf{x}_1$, $\mathbf{x}_2$ respectively. We assume that the photomultipliers are followed by bandpass filters with zero d.c. response, and we shall show that the average value of the correlation between the a.c. fluctuations in the output currents of these filters is identical with that derived above for the radio case by a classical deterministic theory.

As before, we take $J_1(t)$, $J_2(t)$ to be the a.c. output currents of the two filters and we must then calculate $\overline{C}$ the ensemble average of the integrated correlation defined by (2·15) and (2·16).

In this case the calculation is complicated by the necessity of averaging over the number and time of emission of the photoelectrons produced in time T as well as over the radio-frequency phases.

In the development of statistical theory a large number of methods have been evolved for analyzing problems of this kind. The one that we shall use is based on the so-called shot noise representation since, though not perhaps the most elegant procedure, it has the most direct physical interpretation in the present case. This procedure has been extensively studied by Rice (1944) and frequent use will be made of his results.

Following Rice we introduce a normalized probability function $p(t+\mathbf{k}.\mathbf{x}_1)$ such that

$$\int_0^T p(t+\mathbf{k}.\mathbf{x}_1) = 1, \qquad (3\cdot15)$$

which is related to $P(\mathbf{x}_1, t)$ by the equation

$$Np(t+\mathbf{k}.\mathbf{x}) = P(\mathbf{x}_1, t), \qquad (3\cdot16)$$

where

$$N = \sum_{r=1}^{\infty} A\alpha_r n_r = I_0/e \qquad (3\cdot17)$$

and I_0 is the average photoemission current.

We now consider a particular time interval of length T, in which exactly K electrons are emitted from one photocathode and exactly N electrons from the other, so that the output currents of the filters following the photomultipliers may be written

$$J_1(t) = \sum_{k=1}^{K} ef_k(t-t_k), \tag{3·18}$$

$$J_2(t) = \sum_{n=1}^{N} eg_n(t-t_n), \tag{3·19}$$

where $ef_k(t-t_k)$, $eg_n(t-t_n)$ are the effects produced in the first and second filters by electrons of charge e emitted at times t_k and t_n respectively.

Since the filters do not pass d.c.

$$\int_{-\infty}^{\infty} f_k(t-t_k)\,\mathrm{d}t = \int_{-\infty}^{\infty} g_n(t-t_n)\,\mathrm{d}t = 0 \tag{3·20}$$

and it will further be assumed that $f_k(t)$, $g_n(t)$ only differ appreciably from zero in an interval Δ which is negligibly small compared with T.

In a complete discussion it would be necessary to note that $f_k(t)$ will vary from one photoemission to another, since it will depend to some extent upon such things as the emission velocities of the photoelectrons and upon the number and momentum distribution of the secondary electrons produced at each stage of the photomultiplication process. For the present, however, we shall ignore these effects, which would considerably complicate the algebra without adding anything significant to a basic understanding of the phenomenon, though they are of real practical importance since they impose a lower limit to the resolving time of the electronic equipment.

Accordingly, we shall assume that

$$\left.\begin{aligned} f_k(t-t_k) &= f(t-t_k), \\ g_n(t-t_n) &= g(t-t_n). \end{aligned}\right\} \tag{3·21}$$

Finally, we shall limit ourselves to the idealized case

$$f(t) \equiv g(t), \tag{3·22}$$

though to begin with it will be more convenient to retain both symbols.

From (2·15), (3·18) and (3·19) we have that

$$C(T_0) = \frac{1}{T_0}\int_0^{T_0} \mathrm{d}t \sum_{k=1}^{K}\sum_{n=1}^{N} e^2 f(t-t_0-t_{1k})\,g(t-t_{2n}). \tag{3·23}$$

To find $\bar{C}$ we must average over the times of emission of the different photoelectrons, over the total number of photoelectrons emitted in a time interval T, and over the phases of the Fourier components of the radiation field so that

$$\bar{C} = \left\langle \frac{1}{T_0}\int_0^{T_0} \mathrm{d}t \sum_{K=1}^{\infty}\sum_{N=1}^{\infty} e^2 \rho_1(K)\,\rho_2(N) \sum_{k=1}^{K}\sum_{n=1}^{N} \int_0^T f(t-t_0-t_{1k})\,p_1(t_{1k}+\mathbf{k}.\mathbf{x}_1)\,\mathrm{d}t_{1k} \right.$$
$$\left. \times \int_0^T g(t-t_{2n})\,p_2(t_{2n}+\mathbf{k}.\mathbf{x}_2)\,\mathrm{d}t_{2n} \right\rangle_{\text{aver.}}, \tag{3·24}$$

where the angle brackets denote an averaging over the phases of the individual Fourier components of the radiation field. The quantities

$$p_1(t_1+\mathbf{k}.\mathbf{x}_1), \quad p_2(t_2+\mathbf{k}.\mathbf{x}_2)$$

are defined by (3·16), (3·17) and (3·14) and differ only in so far as the areas A_1, A_2 and the position vectors of the photocathodes are not identical. The quantities

$$\rho_1(K), \rho_2(N)$$

are the probabilities that exactly K and N photoelectrons are emitted in time T from the first and second photocathodes respectively. Rice (1944) has discussed the generalization of Campbell's theorem to the case where the probability of a fundamental event varies with time and it can easily be shown, along the lines of his analysis, that

$$\rho_1(K) = \frac{\bar{K}^K \exp(-\bar{K})}{K!}, \quad \rho_2(N) = \frac{\bar{N}^N \exp(-\bar{N})}{N!}, \tag{3·25}$$

where

$$\bar{K} = N_1 T, \quad \bar{N} = N_2 T, \tag{3·26}$$

but this result, which will be needed in the next section, is not essential to the present argument.

If we introduce new time variables τ_{1k}, τ_{2n} defined by

$$\tau_{1k} = t - t_0 - t_{1k}, \quad \tau_{2n} = t - t_{2n} \tag{3·27}$$

in place of $t_{1k} t_{2n}$ we see that

$$\bar{C} = \left\langle \frac{1}{T}\int_0^{T_0} \mathrm{d}t \sum_{K=1}^{\infty}\sum_{N=1}^{\infty} e^2 \rho_1(K)\rho_2(N) \sum_{k=1}^{K}\sum_{n=1}^{N} \int_{-(t-t_0)}^{T-(t-t_0)} f(\tau_{1k})\, p_1(t+\mathbf{k}.\mathbf{x}_1-\tau_{1k})\,\mathrm{d}\tau_{1k} \right.$$
$$\left. \times \int_{-t}^{T-t} g(\tau_{2n})\, p_2(t+\mathbf{k}.\mathbf{x}_2-\tau_{2n})\,\mathrm{d}\tau_{2n} \right\rangle_{\text{aver.}}. \tag{3·28}$$

As long as $t < T - \Delta$ we may replace the integration limits over the variables τ_{1k}, τ_{2n} by $(-\infty, \infty)$ and since Δ/T is, *ex hypothesi*, negligibly small, the resulting error is also negligible.

From (3·16), (3·17) and (3·22) we get that

$$\bar{C} = \left\langle \int_0^{T_0} \frac{\mathrm{d}t}{T_0} e^2 \sum_{K=1}^{\infty}\sum_{N=1}^{\infty} K\rho_1(K)\, N\rho_2(N) \frac{A_2}{A_1} \left\{ \int_0^{\infty} f(\tau_k)\, p_1(t+\mathbf{k}.\mathbf{x}_2-\tau_k)\,\mathrm{d}\tau_k \right\}^2 \right\rangle_{\text{aver.}}. \tag{3·29}$$

From (3·20) only the time-dependent part will contribute to $\bar{C}$, while all the terms explicitly dependent upon the phases of the individual Fourier components of the incident light will average to zero.

Hence, since

$$\left.\begin{aligned} \sum_{K=1}^{\infty} K\rho_1(K) &= \bar{K} = N_1 T = A_1 T \sum_{r=1}^{\infty} \alpha_r n_r, \\ \sum_{N=1}^{\infty} N\rho_2(N) &= \bar{N} = N_2 T = A_2 T \sum_{r=1}^{\infty} \alpha_r n_r, \end{aligned}\right\} \tag{3·30}$$

we have from (3·16), (3·17) and (3·14) that

$$\bar{C} = 2e^2 \sum_{l=L_1}^{L_2} \frac{A_1 A_2 \alpha_l^2 n_l^2}{T} \sum_{m=1}^{M} \frac{|F(fm)|^2}{T}, \tag{3·31}$$

where

$$F(f) = \int_{-\infty}^{\infty} f(t) \exp[-2\pi i f t]\, dt \tag{3·32}$$

is the Fourier transform of $f(t)$ and satisfies the integral equation

$$f(t) = \int_{-\infty}^{\infty} F(f) \exp[2\pi i f t] df \tag{3·33}$$

as long as $\int_{-\infty}^{\infty} |f(t)|\, dt$ exists: a condition which certainly holds in the present case where $f(t)$ satisfies (3·7) and is zero outside $0 < t < \Delta$.

In the limiting case, as $T \to \infty$, (3·31) may be written

$$\bar{C} = 2e^2 A_1 A_2 \int_0^{\infty} \alpha^2(\nu)\, n^2(\nu)\, d\nu \int_0^{\infty} |F(f)|^2 df, \tag{3·34}$$

which is formally identical with the correlation for the radio case given by (2·18), although the physical interpretation of the symbols $\alpha(\nu)$, $F(f)$ is different in the two cases. Thus in the optical case $\alpha(\nu)$ is simply the quantum efficiency of the photocathode, while in the radio case it depends upon the aerial efficiency and the conversion characteristics of the square law detector; again, in the radio case $F(f)$ is simply the frequency characteristic of the filter, while in the optical case it also depends upon the frequency characteristic of the photomultiplier. However, these are minor points which do not alter the fundamental conclusion that the correlation can be found by a purely classical theory in which the photocathode is regarded as a square law detector of suitable conversion efficiency. It is this result which provides the basis for our claim that the correlation is essentially an interference effect exemplifying the wave rather than the corpuscular aspect of light.

(c) *The mean square fluctuations in the emission current of a phototube*

We have argued above that the excess photon noise and the correlation between photons in coherent beams are closely related, and to bring this out more explicitly we shall derive an expression for the mean square fluctuations in the output current of a bandpass filter following a phototube of cathode area A placed normal to an incident plane wave of light.

Let $J_K(t)$ be the output current in the case when exactly K photoelectrons are emitted in time T, then we may use the same shot noise representation in the previous section and write

$$J_K(t) = \sum_{k=1}^{K} ef(t-t_k), \tag{3·35}$$

so that

$$J_K^2(t) = \sum_{k=1}^{K} e^2 f^2(t-t_k) + 2 \sum_{k>n}^{K} \sum_{n=1}^{K} e^2 f(t-t_k) f(t-t_n). \tag{3·36}$$

The mean square fluctuations $\overline{J^2}(t)$ can then be found by averaging over the times of emission of different photoelectrons, over the total number of electrons emitted in time T and over the phases of the Fourier component of the incident light, so that we may write

$$\overline{J^2}(t) = \overline{j_N^2} + \overline{j_C^2}, \tag{3·37}$$

where

$$\overline{j_N^2} = \left\langle \sum_{K=1}^{\infty} e^2 \rho(K) \int_0^T f^2(t-t_k)\, p_1(t+\mathbf{k}.\mathbf{x}_1)\, \mathrm{d}t_k \right\rangle_{\text{aver.}}, \tag{3·38}$$

$$\overline{j_C^2} = \left\langle 2 \sum_{k>n}^{K} \sum_{n=1}^{K} \int_0^T f(t-t_k)\, p_1(t+\mathbf{k}.\mathbf{x}_1)\, \mathrm{d}t_k \int_0^T f(t-t_n)\, p_1(t+\mathbf{k}.\mathbf{x}_1)\, \mathrm{d}t_n \right\rangle_{\text{aver.}}, \tag{3·39}$$

where $p(t+\mathbf{k}.\mathbf{x}_1)$ is defined by (3·14) and (3·16) and $\rho(K)$ is given by (3·25).

We shall now show that $\overline{j_N^2}$, the shot noise contribution to the mean square fluctuations, and $\overline{j_C^2}$ the wave interaction noise, are given by expressions formally identical with those derived in § 2 for the classical radio case.

Only the time *independent* part of $p(t+\mathbf{k}.\mathbf{x}_1)$ contributes to $\overline{j_N^2}$, since the time dependent part depends linearly on the random phases of the Fourier components of the incident light wave and averages to zero.

As long as $t_k < T - \Delta$ we may replace the limits of integration $(0, T)$ by $(-\infty, \infty)$ and, using Parseval's theorem in the form

$$\int_{-\infty}^{\infty} f^2(t)\, \mathrm{d}t = \int_{-\infty}^{\infty} |F^2(f)|\, \mathrm{d}f = 2\int_0^{\infty} |F^2(f)|\, \mathrm{d}f, \tag{3·40}$$

where $f(t)$, $F(\nu)$ are Fourier mates related by (3·32), (3·33), we get that

$$\overline{j_N^2} = 2e^2 A \int_0^{\infty} \alpha(\nu)\, n(\nu)\, \mathrm{d}\nu \int_0^{\infty} |F(f)|^2\, \mathrm{d}f \tag{3·41}$$

on substituting in (3·39) from (3·14), (3·16) and (3·25).

Since J_0, the average emission current of the photocell, is given by

$$J_0 = eA \int_0^{\infty} \alpha(\nu)\, n(\nu)\, \mathrm{d}\nu, \tag{3·42}$$

which is formally identical with (2·20), we see that j_N is indeed the shot noise current, for which

$$\overline{j_N^2} = 2e\, J_0 \int_0^{\infty} |F(f)|^2\, \mathrm{d}f. \tag{3·43}$$

On the other hand, only the time *dependent* part of $p(t+\mathbf{k}.\mathbf{x}_1)$ contributes to $\overline{j_C^2}$, the contribution from the time-independent part being zero from (3·20). By a discussion along identical lines with that given in the previous section it can be shown that

$$\overline{j_C^2} = 2e^2 \int_0^{\infty} A^2 \alpha^2(\nu)\, n^2(\nu)\, \mathrm{d}\nu \int_0^{\infty} |F(f)|^2\, \mathrm{d}f \sum_{K=1}^{\infty} \frac{K(K-1)\rho(K)}{\overline{K}^2}. \tag{3·44}$$

But from (3·25) it follows immediately that

$$\sum_{K=1}^{\infty} \frac{K(K-1)\rho(K)}{\overline{K}^2} = 1, \tag{3·45}$$

so that

$$\overline{j_C^2} = 2e^2 \int_0^{\infty} A^2 \alpha^2(\nu)\, n^2(\nu)\, d\nu \int_0^{\infty} |F(f)|^2\, \mathrm{d}f \tag{3·46}$$

If (3·46) is compared with (3·34) it will be seen that the two expressions are identical in the special case $A_1 = A_2$, which establishes the close connexion between the excess photon noise in a coherent beam of light and the correlation between photons in two coherent light beams. It may also be seen that the expression given by (3·46) is identical with the second term in (2·21) which gives the wave interaction noise for the classical case, so that the excess photon noise due to the so-called 'bunching' of photons is the equivalent, in the corpuscular language, of the wave interaction noise of the undulatory picture. This identification is supported by the analysis of Kahn (1957) who obtains results identical with ours by a treatment based directly on the particle model of the incident light; but quite a different expression for the excess photon noise has been obtained by Fellgett (1949) and by Clark Jones (1953) who relied on thermodynamical arguments. In our view, however, thermodynamical considerations cannot be applied to the photoelectric effect and for this, and other reasons given in appendix II, we consider that their expression for the excess photon noise is wrong. If we may anticipate results which are to be given in a later paper, we may observe that this conclusion is supported by experimental measurements of the correlation between the fluctuations in separate phototubes. These results indicate that the ratio $\overline{j_C^2}/\overline{j_N^2}$ is approximately proportional to α, the quantum efficiency of the photocathodes, which is in accordance with the theory given above, but is incompatible with that of Fellgett (1949) in which $\overline{j_C^2}/\overline{j_N^2}$ should be independent of α.

Ideally, it would be desirable to confirm this conclusion by a direct measurement of the noise in the photoemission current, but this would be very difficult in practice since the excess photon noise is so small by comparison with the shot noise proper. Thus let us consider the case, appropriate to the laboratory experiment reported elsewhere (Hanbury Brown & Twiss 1956*a*), in which the light source is square in shape and subtends an angle θ^2 at the photocathode. Let us assume the idealized conditions in which the radiant energy is linearly polarized and concentrated into a narrow frequency band of rectangular shape centred at 4400 Å with an effective black-body temperature of 7000° K, and that the photocathode has a quantum efficiency of 20 % and a square aperture of width d. If the incident light is to be effectively a plane wave, the source must be so distant that it is not appreciably resolved by the photocathode and this sets an upper limit to the product θd given by the inequality $\theta d < 0{\cdot}2\lambda$ where $\lambda = 4400$ Å is the mean wavelength of the incident light. The number of quanta n incident on the photocathode in unit frequency bandwidth then obeys the inequality $n < 3{\cdot}7 \times 10^{-4}$ so that

$$\overline{j_C^2}/\overline{j_N^2} < \alpha n < 0{\cdot}74 \times 10^{-4}.$$

Since this is appreciably smaller than the uncertainties in the measurement of the shot noise proper, and since we have assumed conditions exceptionally favourable to the observation of the excess photon noise, we can conclude that the contribution of the latter to the total noise current in a phototube is quite negligible in a practical case.

(*d*) *The signal to noise ratio in a measurement of the correlation*

To complete the fundamental theory we shall calculate the signal to noise ratio in a measurement of the correlation. Thus if S is given by

$$S = \overline{C} = \left\langle \frac{1}{T_0} \int_0^{T_0} J_1(t-t_0)\, J_2(t)\, \mathrm{d}t \right\rangle_{\text{aver.}}, \tag{3·47}$$

and N is the r.m.s. fluctuation in $C(T)$ defined by

$$N^2 = \left\langle \left\{ \frac{1}{T_0} \int_0^{T_0} J_1(t-t_0)\, J_2(t)\, \mathrm{d}t \right\}^2 \right\rangle_{\text{aver.}} - \bar{C}^2, \tag{3·48}$$

we shall calculate the ratio S/N.

As we have just seen, the contribution to N^2 due to the excess photon noise is negligible in comparison with the contribution from the shot noise proper, therefore in finding N we can assume that the fluctuations in the emission currents of the two photocathodes are due to independent shot noise currents. To this order of accuracy $J_1(t-t_0)$, $J_2(t)$ may be represented by the Fourier series

$$\left.\begin{aligned} J_1(t-t_0) &= \sum_{n=1}^{\infty} \gamma_n \cos\left(\frac{2\pi nt}{T} - \phi_n\right), \\ J_2(t) &= \sum_{n=1}^{\infty} \eta_m \cos\left(\frac{2\pi mt}{T} - \psi_m\right), \end{aligned}\right\} \tag{3·49}$$

where ϕ_n, ψ_m are independent random variables distributed with uniform probability over the range 0 to 2π.

If the photomultiplying process and the bandpass filters introduce no additional noise it follows immediately from (3·43), with $\mathrm{d}\nu = 1/T$ that

$$\left.\begin{aligned} \tfrac{1}{2} j_N^2 &= 2eI_1 \frac{|F_1^2(fn)|}{T}, \\ \tfrac{1}{2}\eta_m^2 &= 2eI_2 \frac{|F_2^2(f_m)|}{T}, \end{aligned}\right\} \tag{3·50}$$

if the amplitude and phase response of the photomultiplier are included in $F(f)$. When the gain M of the photomultiplier is large, the noise introduced by the bandpass filters is normally negligible; but the number of secondary electrons emitted at a given stage of the photomultiplier is itself a fluctuating quantity, and it has been shown by Shockley & Pierce (1938) that this effect increases the output noise power by a term

$$\frac{M\mu - 1}{M(\mu-1)} \simeq \frac{\mu}{\mu - 1}, \tag{3·51}$$

if $M \gg 1$, where μ is the secondary emission multiplication factor.

It is therefore more realistic to assume that

$$\left.\begin{aligned} \tfrac{1}{2} j_n^2 &= 2eI_1 \frac{\mu}{\mu-1} \frac{|F_1(f_n)|^2}{T}, \\ \tfrac{1}{2}\eta_m^2 &= 2eI_2 \frac{\mu}{\mu-1} \frac{|F_2(f_m)|^2}{T}. \end{aligned}\right\} \tag{3·52}$$

From (3·49) we have immediately that

$$N^2 = \left\langle \left| \frac{1}{T_0} \int_0^{T_0} \mathrm{d}t \sum_{n,m=1}^{\infty} \gamma_n \eta_m \cos(2\pi f_n t - \phi_n) \cos(2\pi f_m t - \psi_m) \right|^2 \right\rangle_{\text{aver.}}, \tag{3·53}$$

where the angle brackets now mean that the expression contained within them is to be averaged over the random variables ϕ_n, ψ_m; since these phases are all mutually uncorrelated only terms independent of them contribute to N^2.

Integrating over time we get that

$$N^2 = \left\langle \left| \sum_{n,m=1}^{\infty} \frac{\gamma_n \eta_m}{2T_0} \left[\frac{\sin\{2\pi(f_n - f_m)\,T_0 - (\phi_n - \psi_m)\} + \sin(\phi_n - \psi_m)}{2\pi(f_n - f_m)} \right.\right.\right.$$
$$\left.\left.\left. + \frac{\sin\{2\pi(f_n + f_m)\,T_0 - (\phi_n + \psi_m)\} + \sin(\phi_n + \psi_m)}{2\pi(f_n + f_m)} \right] \right|^2 \right\rangle_{\text{aver.}}. \tag{3·54}$$

If we collect the terms independent of the random phases and proceed to the limit in which sums are replaced by integrals we get from (3·40) that

$$N^2 = \left(\frac{2e\sqrt{(I_1 I_2)}\,\mu}{\mu - 1}\right)^2 \frac{1}{T_0} \int_0^\infty |F_1(f_1)|^2 \,\mathrm{d}f_1 \int_0^\infty |F_2(f_2)|^2 \,\mathrm{d}f_2$$
$$\times \frac{1}{2}\left[\frac{\sin^2\pi(f_1+f_2)\,T_0}{\pi^2(f_1+f_2)^2} + \frac{\sin^2\pi(f_1-f_2)\,T_0}{\pi^2(f_1-f_2)^2}\right], \tag{3·55}$$

a result very similar to that derived, for a somewhat different case, by Rice (1945). The contribution to N^2 from the term proportional to

$$\sin^2(\pi(f_1+f_2)\,T_0)/\pi^2(f_1+f_2)^2$$

is quite negligible in the practical case where the integration time $T_0 > 10^3$ s, and where the lowest frequency passed by the filter following the photomultiplier tube is $> 10^6$ c/s. Furthermore, with $T > 10^3$, then $F(f_1) \simeq F(f_2)$ for values of $f_1 - f_2$ for which $\sin^2(\pi(f_1-f_2)T_0)/\pi^2(f_1-f_2)^2$ differs significantly from zero, and in this case N^2 may be written in the simplified form

$$N^2 = \left(\frac{2e\sqrt{(I_1 I_2)}\,\mu}{\mu - 1}\right)^2 \frac{1}{T_0} \int_0^\infty |F_1(f)\,F_2(f)|^2 \,\mathrm{d}f \,\frac{1}{2\pi} \int_0^\infty \frac{\sin^2 X}{X^2}\,\mathrm{d}X. \tag{3·56}$$

The effective bandwidth of the bandpass filters, which we now assume to be identical, may be defined by the expression

$$b_v = \int_0^\infty |F(f)|^2 \,\mathrm{d}f / F^2_{\text{max.}}, \tag{3·57}$$

where
$$|F_1^2(f)| = |F_2^2(f)| = |F(f)|^2$$

and $F_{\text{max.}}$ is the maximum value of $F(f)$.

If we define a normalized spectral density coefficient η by the relation

$$\eta = \int_0^\infty |F^4(f)|\,\mathrm{d}f \Big/ F^2_{\text{max.}} \int_0^\infty |F^2(f)|\,\mathrm{d}f, \tag{3·58}$$

then, since
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2 X}{X^2}\,\mathrm{d}X = \tfrac{1}{2}, \tag{3·59}$$

we get from (2·20), (3·56) and (3·57) that†

$$N = \frac{\sqrt{2e^2\mu}}{\mu - 1}(A_1 A_2)^{\frac{1}{2}} (b_v/T_0)^{\frac{1}{2}} \eta^{\frac{1}{2}} \mid F_{\text{max.}} \mid^2 \int_0^\infty \alpha(\nu)\, n(\nu)\, d\nu. \tag{3·60}$$

From (3·34) and (3·60) we have that the signal to noise ratio S/N is given by

$$\left(\frac{S}{N}\right)_{\text{r.m.s.}} = \sqrt{2}\,(1 - 1/\mu)\,(A_1 A_2)^{\frac{1}{2}} (b_v T_0)^{\frac{1}{2}} \eta^{-\frac{1}{2}} \int_0^\infty \alpha^2(\nu)\, n^2(\nu)\, d\nu \bigg/ \int_0^\infty \alpha(\nu)\, n(\nu)\, d\nu, \tag{3·61}$$

which is independent of $\mid F_{\text{max.}} \mid^2$ and, therefore, of the gain of the photomultiplier tubes.

This result has been derived for the case where the incident light is linearly polarized. If the light is unpolarized, as is normally the case, the expression for N will be unaltered if the average number of quanta per cycle bandwidth is unaltered. However, the expression for S will be different since there is no correlation between quanta in different states of polarization, and we must decompose the incident beam into two independent components each with $\frac{1}{2}n$ quanta per cycle bandwidth. Since the correlation between coherent beams of polarized light is proportional to the square of the number of quanta per cycle bandwidth, the value for S must be reduced by a factor $\frac{1}{2}$, and the signal to noise ratio for the case of unpolarized light becomes

$$\left(\frac{S}{N}\right)_{\text{r.m.s.}} = (1/\sqrt{2})\,(1 - 1/\mu)\,(A_1 A_2)^{\frac{1}{2}} (b_v T_0)^{\frac{1}{2}} \eta^{-\frac{1}{2}} \int_0^\infty \alpha^2(\nu)\, n^2(\nu)\, d\nu \bigg/ \int_0^\infty \alpha(\nu)\, n(\nu)\, d\nu. \tag{3·62}$$

Because of the very small number of quanta received per unit bandwidth from even the hottest sources the signal to noise ratio will only be significant if one integrates for long times and also accepts the intensity fluctuations over the widest possible bandwidth. However, the signal to noise ratio is independent of the bandwidth of the incident light.

4. Discussion

In calculating the correlation between the emission times of photoelectrons at different points on the wavefront of a plane wave of light we have used a quantum theory in which the incident radiation field is treated classically. Such a procedure is justified theoretically by the fact that the relevant observables of the radiation field can be characterized by commuting operators, but it is opposed to one's natural tendency to regard a correlation between the emission times of photoelectrons as essentially a quantum phenomenon for which a classical treatment of the radiation field would only be valid in the limiting case where the number of incident quanta is very large. Accordingly, to make the argument more acceptable from a physical point of view we shall consider the analogous case of a diffraction grating illuminated by monochromatic light to produce an interference pattern on a screen.

In this latter case it is well known that the average distribution of light intensity over the screen can be found by a classical wave theory, even in the limiting case

† This expression for N is smaller by a factor $1/\sqrt{2}$ than that given in an earlier paper (Hanbury Brown & Twiss 1956*a*) which applies when the correlation is measured by a somewhat different technique.

where the light is so weak that one can count the arrival of individual photons. As Born (1945) points out in discussing an essentially similar experiment, things are in no way altered if the screen is replaced by a mosaic of photoelectric elements; the experiment still illustrates the wave aspect of light, since the particle aspect can only really be brought out by observations in which the position of a single quantum is measured at two successive instants of time. If such observations were introduced into the present experiment, the interference pattern would be destroyed.

From the point of view in which light is viewed as a photon stream the appearance of interference effects is closely related to Heisenberg's uncertainty principle; an accurate knowledge of the transverse point of impact of a photon involves a corresponding uncertainty in the transverse momentum, and therefore an uncertainty in the element of the grating from which the photon has come. If we perform an experiment, such as blackening out the rest of the grating to determine the transverse momentum of the photons, then the main interference pattern once more disappears.

This state of affairs is exactly paralleled by the correlation experiment, which is the subject of this paper, if one everywhere substitutes the concepts of time and energy for those of position and momentum. Thus the bandwidth of the light and the length of the observation time in the correlation experiment are the analogues of the width and number of lines per unit length of the grating. The interference pattern in time, the beat phenomenon of the correlation experiment, arises because of the uncertainty in the energy of the photon which produced a specific photoemission, and is the analogue of the interference pattern on the screen which arises because of the uncertainty in the transverse momentum of a photon reaching a specific point on the screen. Both phenomena are to be understood from the particle point of view as being due to an uncertainty in the behaviour of a single photon and not as due to interference between different quanta. Finally, it may be noted, that if one partly removes the uncertainty in the energy of the incident photons in the correlation experiment, by using a highly monochromatic source or by analyzing the light with a prism of very great resolving power, the higher beat frequency components will disappear, just as the analogous components of the interference pattern on the screen will disappear if the angular width of the grating is suitably reduced.

Accordingly, since the radiation field can be treated classically in the case of the diffraction grating, it is only to be expected that it can be treated classically in analyzing the correlation experiment.

In this paper we have considered the idealized case where the two photocathodes lie on the same wavefront of the incident light. However, the emission time of a photoelectron is uncertain within limits determinined in practice by the resolving time of the electronic equipment, so the observed correlation will not be affected as long as the difference in the time of arrival of a particular wavefront of the two photocathodes is small compared with this resolving time. This means that the position of the photocathodes need only be controlled to an accuracy determined by the bandwidth of the fluctuations rather than by the wavelength or bandwidth of the incident light.

To simplify the presentation, we have developed the fundamental theory for the case where this incident radiation is a plane wave. However, an arbitrary radiation field can be expressed as a sum of plane waves and, since the operators associated with the observables of one plane wave commute with all the operators associated with the observables of any other, one is equally justified in analyzing this general case by a theory in which the radiation field is treated classically; this will be done in a subsequent paper.

A quantum theory is needed to compute the probability of photoemission which, as we have shown, is proportional to the square of the amplitude of the incident light; but if this probability is known from experiment, one can calculate the correlation between the fluctuations in the photoemission currents at two separate phototubes by a fully classical theory in which the photocathodes are regarded as square law detectors of a suitable conversion efficiency. This emphasizes the fact that the theory is equally valid if the phototubes are replaced by true energy detectors such as bolometers or thermistors, though for reasons of signal to noise ratio these latter alternatives could not be used in a practical correlation experiment.

A purely classical theory can also be used to calculate the mean square fluctuations in the emission current of a single phototube. As we have shown these fluctuations can be represented as the sum of two terms, a *shot noise* term due to the discrete nature of the electrons carrying the photocurrent, and a term which we have called the *wave interaction noise* because in the classical theory it arises from the beats between the different Fourier components of the radiation field. The expressions for these terms are identical with those derived by Kahn (1957) in a treatment based directly on quantum statistics, and two conclusions can be drawn from this. First, that the shot noise is a consequence of the corpuscular nature of the electrons, it does not depend at all on the fact that the radiation field is also quantized; secondly, that the wave interaction noise is identical with the excess photon noise which is interpreted, in the language of the corpuscular theory, as due to the so-called 'bunching' of photons and which is essentially a consequence of the fact that light quanta obey Bose–Einstein statistics. This so-called 'bunching' is, of course, in no way dependent upon the actual mechanism by which the light energy was originally generated; still less does it imply that the photons must have been injected coherently into the radiation field. On the contrary, if one wishes to picture the electromagnetic field as a stream of photons, one has to imagine that the light quanta redistribute themselves over the wavefront, as the radiation field, which may be quite incoherent in origin, is focused and collimated into beams capable of mutual interference; thus the correlation between photons is determined solely by the energy distribution and coherence of the light reaching the photon detectors.

Appendix I. Coherent interference and the extent in real space of an elementary cell in phase space

When one is dealing with particles such as gas molecules the dimensions, in real space, of an elementary cell of volume L^3 in phase space are likely to be very small. Thus, in the extreme case of a hydrogen gas in which the uncertainties in the momenta are those appropriate to a thermal spread of 1° K, the dimensions, in real space, of the elementary cell are of the order of 10 Å cube and these will be correspondingly reduced for heavier gases or for larger uncertainties in the momenta of the individual molecules.

However, things are quite different in the case of light if the angular size of the source is very small. Thus, when light is received from a star, the volume in real space of an elementary cell can be many cubic metres, and we shall prove the general result that as long as two points are close enough together to permit virtually complete interference between the light rays reaching them, which implies that their separation is insufficient to resolve the star, then they lie in the same elementary cell in phase space.

For simplicity let us consider a very distant source of square angular aperture $\theta \times \theta$, where θ is very small, and two observing points on the earth with relative coordinates (Δx, Δy, Δz) such that the light source lies on the z axis. Let us further suppose that the light is concentrated with a narrow frequency band of width $\Delta\nu$.

Then, since the volume of an elementary cell in phase space is h^3 we have that

$$\Delta p_x \Delta p_y \Delta p_z \Delta q_x \Delta q_y \Delta q_z = h^3, \tag{A 1}$$

where Δp, Δq are the uncertainties in the momentum and position respectively.

In the present case

$$\Delta p_x = \Delta p_y = \frac{h\nu}{c}\theta, \tag{A 2}$$

while if θ^2 is negligible compared with $\Delta\nu/\nu$

$$\Delta p_z = h\Delta\nu/c. \tag{A 3}$$

Substituting from (A 2) and (A 3) in (A 1) we get that

$$\theta^2 \frac{\Delta q_x \Delta q_y}{\lambda^2} \frac{\Delta q_z}{c} \Delta\nu = 1, \tag{A 4}$$

where $\lambda = c/\nu$.

Now if the interference fringes obtained from two coherent beams of light bandwidth $\Delta\nu$ are not to be appreciably weakened, the difference in the path length of the two beams must not exceed a wavelength of a frequency $\Delta\nu$ so that we must have

$$\Delta\nu\Delta z/c < 1. \tag{A 5}$$

Furthermore, if the transverse separation of the two points is to be so small that they do not resolve the source, one must certainly have

$$\theta\Delta x/\lambda < 1, \quad \theta\Delta y/\lambda < 1. \tag{A 6}$$

Combining the inequalities (A 5) and (A 6) with (A 4) one gets

$$\Delta x \Delta y \Delta z < \Delta q_x \Delta q_y \Delta q_z, \tag{A 7}$$

and since $\Delta q_x \Delta q_y \Delta q_z$ is the spatial volume of the elementary cell in phase space we have proved the required result.

The importance of this argument from the theoretical point of view is that it brings out the connexion between the wave and particle interpretations of the phenomenon intensity interference. Thus, on the classical picture one would expect the intensity fluctuations in the light at two different points in space to be correlated as long as the light rays reaching these two points were capable of mutual interference; while on the quantum picture one would expect a correlation between the arrival times of quanta at different points as long as these lie in the same cell in phase space, and the above discussion shows that if the latter condition is satisfied then so is the former.

Appendix II. On the excess photon noise of light detectors

In the text we derived an expression for the noise in a photoemission current from first principles, and noted that an identical result has been obtained by Kahn (1957) with the aid of quantum statistics. However, a quite different result has been given by Fellgett (1949) and also by Clark Jones (1953) from thermodynamical arguments, and in this appendix we give the reasons for rejecting their procedure.

The thermodynamical argument depends in the first place on an analysis of a thermal detector in thermal equilibrium with a blackbody enclosure at temperature T. The discussion given by Clark Jones is based on a general theorem by Callen & Welton (1951) which enables one to find the fluctuations associated with a linear dissipation process, and which represents a powerful generalization of Nyquist's theorem (1928) to cover any case where the underlying physical process can be characterized by a generalized impedance. The treatment by Fellgett is more specific in that the equivalent electrical circuit is explicitly derived for a given detector, but it is identical in essentials. Both writers assume that the fluctuations in the thermal detector output are equal to the energy fluctuations in the thermal radiation field, half being due to the absorbed and half to the emitted radiation. In the case of the photocell the emitted stream of radiation does not exist so, it is argued, the fluctuations in this case will be reduced by one-half.

Two objections to this treatment are immediately apparent. In the first place the theorem of Callen & Welton does not apply, since the dissipation process of a thermal detector is non-linear, the equivalent resistance being itself a function of the temperature. Admittedly if the temperature fluctuations are very small compared with T the error is also small, but then so is the contribution of the excess photon noise. If we consider the analogous case of a radio antenna in a blackbody of temperature T, then the voltage fluctuations across the output terminals of the antenna can be found from Nyquist's theorem by taking the radiation resistance of the antenna to be at temperature T. However, if a square-law detector were placed between the antenna and the output terminals, one could not use a generalized Nyquist's theorem to find the energy fluctuations in the incident field or the fluctuations across the output terminals of the square-law detector, since this would ignore the presence of beats between different components of the radiation field.

The second and more serious objection is that one cannot in general equate the fluctuations in the output of a thermal or photon detector to the energy fluctuations in the thermal radiation field; the principle of detailed balancing applies to the average flow of energy but not to the fluctuations themselves. It is essentially for this reason that the analysis for the thermal detector cannot be applied to the photocell and that the estimate of the excess photon noise given by Fellgett and Clark Jones is linearly proportional to the quantum efficiency rather than quadratically proportional as found by Kahn and by the present writers.

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