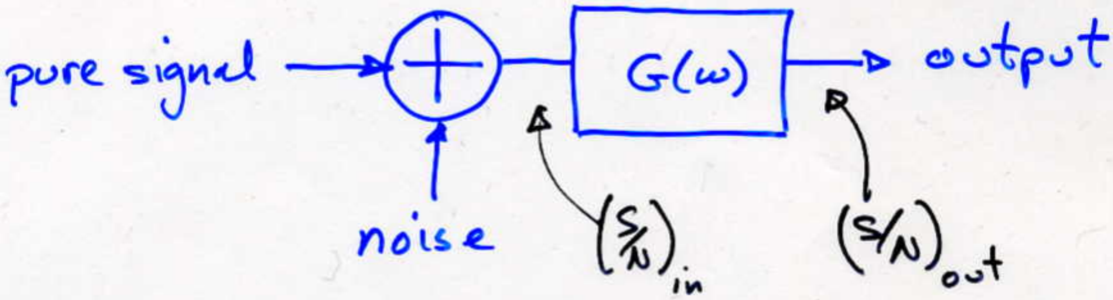
Why Filter?

Consider a generalized detection process with imperfect response $G(\omega)$ and with noise. The result is an apparent signal corrupted with noise and smeared.



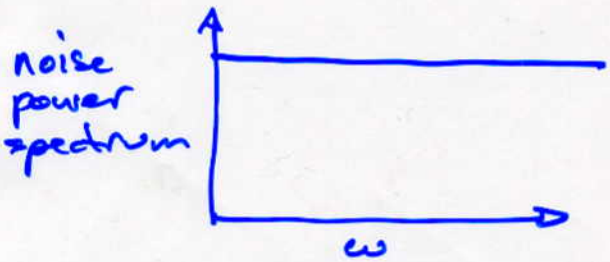
The goal is to estimate or "reconstruct" the pure signal. In some cases the form of the signal is <u>known</u> and we merely need to estimate its arrival time (or phase). Obviously, the <u>output</u> must be processed in some way which is optimal for the solution of the problem.

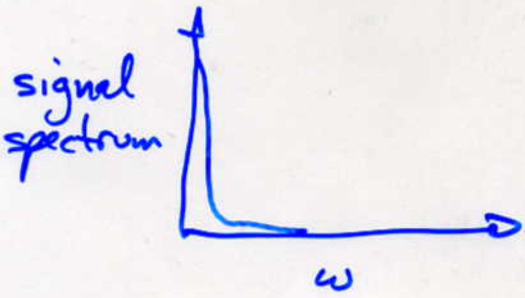
This is called a "filter" when it is done before data sampling + recording, although post-recording digital "filters" also exist. The purpose of these "post-filters" ~~is~~ is usually very different than "pre-filters".

Why not just record everything, and then filter later?

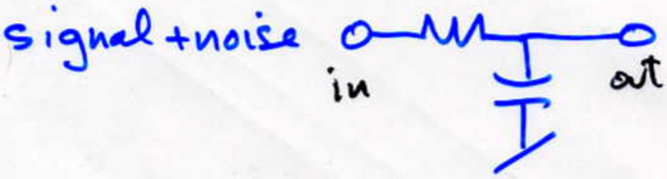
Good idea. Unfortunately real systems can loose information due to A/D conversion and digital sampling. So when there is wide-spectrum noise some filtering is needed.

Trivial example: Your signal is a slowly varying voltage and the noise is large and extends to arbitrarily high frequencies. You are using a sampling data recording system with sample time T_{samp} shorter than characteristic times in the signal $s(t)$.





Use a low-pass filter:

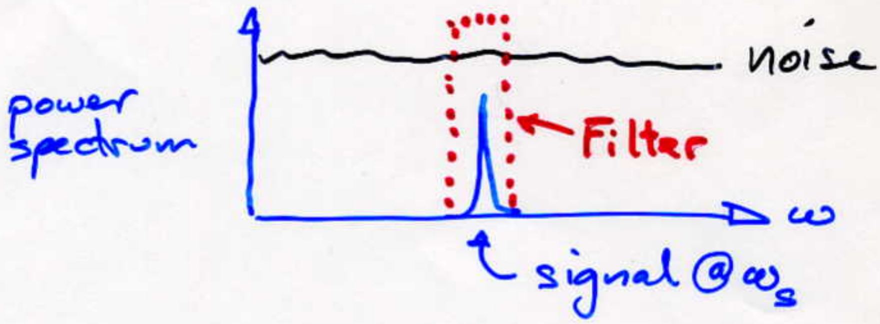


$RC = \tau > T_{samp}$

This is an example of a broad class of filter applications where the signal and noise have mainly different spectra.

Another trivial example:

Your signal is at some known frequency and is slowly varying, in the presence of wide-band noise:



The S/N ratio is now higher at the output of this filter, since no signal Fourier components are lost but most of the noise is.

Caution: Suppose there were an unexpected additional signal at $\omega_{s2} \neq \omega_s$ outside this filter bandpass. This new signal at ω_{s2} would be lost, including any related new science!

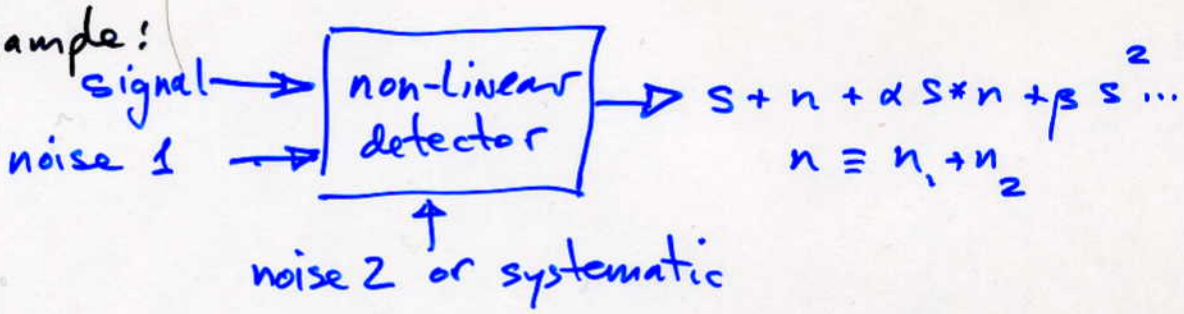
Lesson: DO NO HARM

The process of discovery can be damaged by over-filtering.

- Filter out only known noise at frequencies where no signals are likely present.
- Filter out known sources of systematics + noise which cause system response to degrade.

Avoiding unwanted system responses, which can <u>mix</u> signal and noise in new ways, is a <u>major</u> application of filtering. i.e. real detectors and systems are not perfect, and the $(S/N)_{OUT}$ can be less than $(S/N)_{in}$ if unnecessary noise and other unwanted signals are allowed to pass through.

Example:



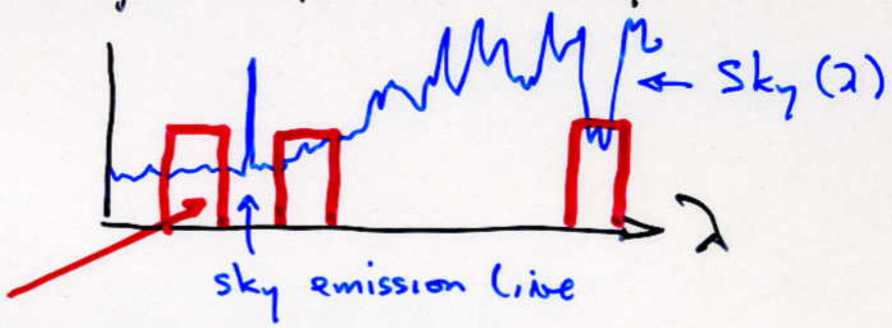
Most systems and detectors have responses which are understood and Linear only over some finite "Dynamic Range".

- Keep system within its dynamic range by pre-filtering.

Pre-filtering is done at the <u>input</u> signal and noise frequencies. Multiple filters are often necessary.

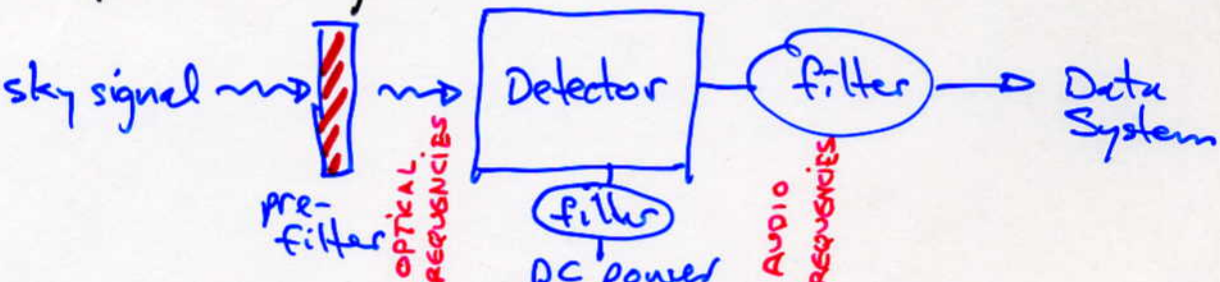
Example 1: LIGO. Seismic vibrations must be filtered out before they enter the interferometer test mass. Laser noise must be filtered out before light enters interferometer...

Example 2: Ground-based photometry of stars and galaxies. Source optical spectrum $S(\lambda)$. Sky background spectrum $Sky(\lambda)$.



Design pre-filters which avoid wavelengths where the background is high or variable.

In general, the output of the detector is also filtered to remove noise. And power sources to the detector must be filtered to remove spurious signals:



However, TOO MUCH "SIGNAL" FILTERING CAN LOSE USEFUL DATA FOREVER.

- Do no harm. Filter only when and where necessary to maintain linearity, S/N, ...
- You can always then re-analyze your data with new digital post-filters!

Types of Filters

There exists a filter for every application.

Two broad classes:

- Signal(s) are exactly known, and we need to know their presence or arrival time.

  Examples: Radar, communications.

- Only statistical information is known about the signal and the noise. (Most other applications). Examples: Neural nets, pre- and post-filters for $(S/N)_{MAX}$ ...

For each of these classes there are many possible approaches (or types) for filtering depending on the application:

- Invariant. Recursive. Adaptive.
- Linear. Non-Linear.

Applications: Minimize the effects of systematics. Maximize S/N. Deconvolution. Minimize square error. Signal recovery...

The detailed form of the "optimal" filter depends on the application as well as what is known about the signal, noise, and system response.

Linear Filters

$$s_1(t) + s_2(t) \longrightarrow \boxed{\text{Filter}} \longrightarrow s_1 * h + s_2 * h$$

response $h(t)$

Example application from radar or binary communication:

The set of possible signals is precisely known $s_1(t)$, $s_2(t)$ ... Let the received signal be some unknown combination of $s_i(t+\tau_j)$ arriving at unknown times τ_j.

The so-called "correlator receiver" [remember the PSD?] with integration time T:

$$r(t) \longrightarrow \underset{\underset{s_1(t)}{\uparrow}}{\otimes} \longrightarrow \boxed{\int_0^T} \longrightarrow e_1(T) = \int_0^T r(t)\, s_1(t)\, dt$$

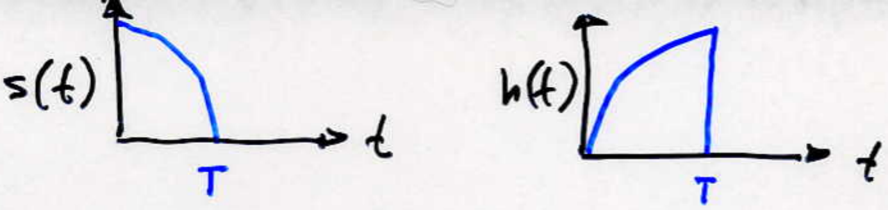
Can $e_1(T)$ be obtained via some linear filter? If linear filter weight is $h_1(t)$, output is $e_1(t) = \int_0^t h_1(\tau)\, r(t-\tau)\, d\tau$. Output at time T is $e_1(T) = \int_0^T h_1(\tau)\, r(T-\tau)\, d\tau$. Make fortuitous selection

$$h_1(t) = s_1(T-t), \quad 0 \le t \le T$$

$$\Rightarrow e_1(T) = \int_0^T s_1(T-\tau)\, r(T-\tau)\, d\tau = \int_0^T s_1(t)\, r(t)\, dt$$

note

So-called <u>signal-matched</u> filter.



The matched filter has the same form as the signal but reversed in time. Note that the output of the correlator receiver and the output of the matched filter are equal only at the sample time T.

This result is also true for additive white noise. If the noise is <u>not</u> white, a more general filter can be derived which maximizes signal/noise:

$$\int_0^T h(z)\, C_N(\tau - z)\, dz = s(T-\tau), \quad 0 \le \tau \le T$$

where $C_N \equiv \langle n(t)\, n(t-\tau) \rangle$ is the auto-correlation function of the noise.

No Gaussian assumption was used!

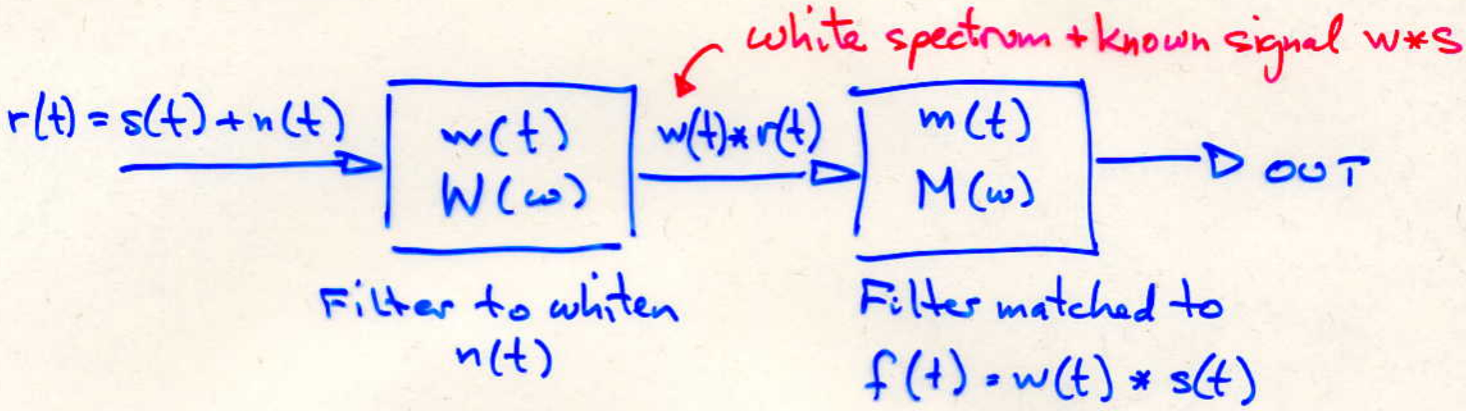
Friedholm integral equation of 1^{ST} kind.

Approximate solution:

$$\int_{-\infty}^{\infty} h_0(z)\, C_N(\tau - z)\, dz = s(T-\tau), \quad -\infty < \tau < \infty$$

Fourier transform: $$H_0(\omega) = \frac{\tilde{S}^*(\omega)\, e^{-i\omega T}}{S_N(\omega)}$$

We can see this solution via a more intuitive approach using a "pre-whitening" filter.



i.e. filter first to make the noise white, then you can do a matched filter.

Single equivalent filter: $H(\omega) = W(\omega)\, M(\omega)$

Choose $M(\omega)$ to maximize S/N ratio at some time T: $M(\omega) = F^*(\omega)\, e^{i\omega T}$

where $F(\omega)$ is Fourier transform of $s(t) * w(t)$.

Theorem: $$H(\omega) = \frac{\tilde{S}^*(\omega)\, e^{-i\omega T}}{S_N(\omega)}$$

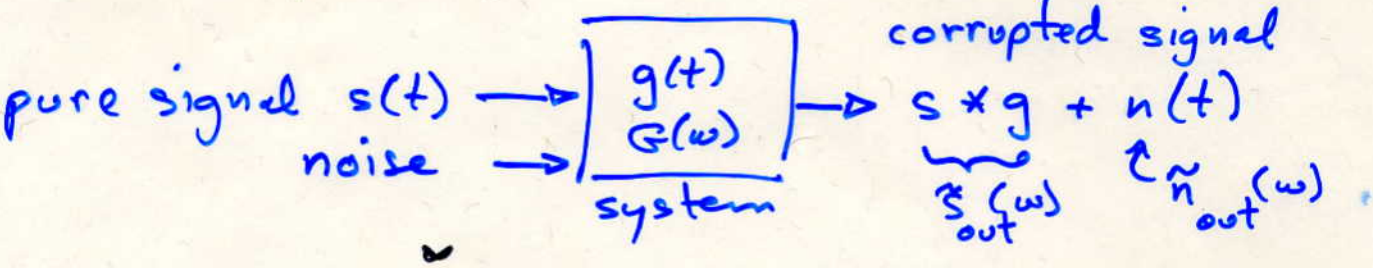
FT of time-reversed sig. ← noise power spectral density

i.e. for "red" noise

the optimal filter de-weights low frequencies, relying more on the high frequency components of $S(\omega)$.
Low frequency information in $S(\omega)$ becomes less useful due to the increased noise, in this example.

Wiener Filter (simple linear filter example)

Minimize statistical difference between optimal <u>linear</u> filter output and the original uncorrupted signal.



$$s(t) * g(t) = \int_{-\infty}^{\infty} g(t-\tau)\, s(\tau)\, d\tau \qquad \widetilde{s * g}\,(\omega) = G(\omega)\, \tilde{s}(\omega)$$

In the case of zero noise, the optimal filter is simply $G^{-1}(\omega)$. "Deconvolution".

<u>With</u> noise, consider optimal filter $H(\omega)$

such that $H(\omega)\left[\widetilde{s * g} + \tilde{n}\right] \equiv \tilde{s}_0(\omega)$

where we minimize $\int_{-\infty}^{\infty} \left|\tilde{s}_0(\omega) - \tilde{s}(\omega)\right|^2 d\omega$.

Sol$^{\underline{n}}$:

$$H(\omega) = \frac{\left|\tilde{s}_{out}(\omega)\right|^2}{\left|\tilde{s}_{out}(\omega)\right|^2 + \left|\tilde{n}_{out}(\omega)\right|^2}$$

Note $H \to 1$ where noise is small,

$H \to 0$ where noise is large.

Non-iterative. Gaussian noise and signal.

The Wiener filter can be implemented (or designed) by simply "looking" at the output and noting the regions of the spectrum dominated by signal. But if S<<N, then one must know or guess the signal spectrum. Assumes signal and noise are Gaussian and stationary — otherwise it is not an optimal linear filter.

REALLY DUMB FILTER!

Other, non-linear or iterative, filters can incorporate more priors and are more optimal in non-Gaussian or non-stationary data.

Iterative filters

- Quantify goodness of signal reconstruction
- Iterate until maximally acceptable

Generalized recursive filter: