Johnson Noise and Shot Noise: The Determination of the Boltzmann Constant, Absolute Zero Temperature and the Charge of the Electron

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Some kinds of electrical noise that interfere with a measurement may be avoided by proper attention to experimental design such as provisions for adequate shielding or multiple grounds. Johnson noise [1] and shot noise, however, are intrinsic phenomena of all electrical circuits, and their magnitudes are related to important physical constants. The purpose of this experiment is to measure these two fundamental electrical noises. From the measurements, values of the Boltzmann constant, k, and the charge of the electron, e will be derived.

PREPARATORY QUESTIONS

1. Define the following terms: Johnson noise, shot noise, RMS voltage, thermal equilibrium, temperature, Kelvin and centigrade temperature scales, entropy, dB.
2. What is the Nyquist theory **??** of Johnson noise?
3. What is the mean square of the fluctuating component of the current in a photodiode when its average current is I_{av} ?

The goals of the present experiment are:

1. To measure the properties of Johnson noise in a variety of conductors and over a substantial range of temperature and to compare the results with the Nyquist theory;
2. To establish the relation between the Kelvin and centigrade temperature scales,
3. To determine from the data values for the Boltzmann constant, k, and the centigrade temperature of absolute zero.

INTRODUCTION

> "Thermal physics connects the world of everyday objects, of astronomical objects, and of chemical and biological processes with the world of molecular, atomic, and electronic systems. It unites the two parts of our world, the microscopic and the macroscopic."[4]

By the end of the 19th century the accumulated evidence from chemistry, crystallography, and the kinetic theory of gases left little doubt about the validity of the atomic theory of matter, though a few reputable scientists still argued strongly against it on the grounds that there was no "direct" evidence of the reality of atoms. In fact there was no precise measurement yet available of the quantitative relation between atoms and the objects of direct scientific experience such as weights, meter sticks, clocks, and ammeters.

To illustrate the dilemma faced by physicists in 1900, consider the highly successful kinetic theory of gases based on the atomic hypothesis and the principles of statistical mechanics from which one can derive the equipartition theorem. The theory showed that the well-measured gas constant R_g in the equation of state of a mole of a gas at low density,

$$PV = R_g T \tag{1}$$

is related to the number of degrees of freedom of the system, $3N$, by the equation

$$R_g = \frac{k(3N)}{3} = kN \tag{2}$$

where N is the number of molecules in one mole (Avogadro's number), and k is the Boltzmann constant defined so that the mean energy per translational degree of freedom of the molecules in a quantity of gas in thermal equilibrium at absolute temperature T is $kT/2$. At the turn of the century nobody knew how to measure precisely either k or N. What was required was either some delicate scheme in which the fundamental granularity of atomic phenomena could be detected and precisely measured above the smoothness that results from the huge number of atoms in even the tiniest directly observable object, or a thermodynamic system with a measurable analog of gas pressure and a countable number of degrees of freedom.

The Millikan oil drop experiment of 1910 was a delicate scheme by which the quantum of charge was accurately measured. It compared the electrical and gravitational forces on individual charged oil droplets so tiny that the effect of a change in charge by one or a few elementary charges could be directly seen and measured through a microscope. The result was a precise determination of e which could be combined with the accurately known values of various combinations of the atomic quantities such as the faraday (N_e), e/m , atomic weights, and the gas constant (kN), to obtain precise values of N, k, and other atomic quantities. Therefore, a current will not be continuous in the mathematical sense, it should exhibit a "noise" due to the granularity of charges.

Twenty years later Johnson discovered an analog of gas pressure in an electrical system, namely, the mean square "noise" voltage across a conductor due to thermal agitation of the electrical modes of oscillation which are coupled to the thermal environment by the charge carriers. Nyquist showed how to relate that mean square voltage to the countable number of degrees of freedom of electrical oscillations in a transmission line. The only atomic constant that occurs in Nyquist's theoretical expression for the Johnson noise voltage is the Boltzmann constant k. A measurement of Johnson noise therefore yields directly an experimental determination of k.

states in the solar corona.

In classical statistical mechanics, $k/2$ is the constant of proportionality between the Kelvin temperature of a system in thermal equilibrium and the average energy per dynamical degree of freedom of the system. Its ultimate quantum physical significance emerged only with the development of quantum statistics after 1920 [4]. A summary of the modern view is given below. (see Kittel and Kroemer for a lucid and complete exposition):

Entropy and Temperature

A closed system of many particles exists in a number of distinct quantum states consistent with conservation constraints of the total energy of the system and the total number of particles. For a system with g accessible states, the fundamental entropy σ is defined by

$$\sigma = \ln g \tag{3}$$

With the addition of heat the number of states accessible within the limits of energy conservation rises, and the entropy increases. An exact enumeration of the quantum states accessible to a system composed of many non-interacting particles in a box and having some definite energy can be derived from an analysis based on the solutions of the Schrödinger equation (Ref. [4], p 77). It shows that for one mole of a gas at standard temperature and pressure (273K, 760 mm Hg) σ is of the order of 10^{25}. The corresponding value of g is of the order of the huge number $e^{10^{25}}$!

Suppose that the total energy U of the system is increased slightly by ΔU, perhaps by the addition of heat, while the volume and number of particles are held constant. With the increase in energy more quantum states become accessible to the system so the entropy is increased by $\Delta\sigma$. The fundamental temperature is defined by

$$\frac{1}{\tau} \equiv \left(\frac{\partial \sigma}{\partial U}\right)_{N,V} \tag{4}$$

The units of τ are evidently the same as those of energy. Since an increase in the energy of one mole of gas by one joule causes a very large increment in σ , the magnitude of τ in common circumstances like room temperature must be much less than 1. In practical thermometry, the Kelvin temperature T is proportional to τ, but its scale is set by defining the Kelvin temperature of the triple point of water to be exactly 273.16 K. This puts the ice point of water at 273.15 K and the boiling point 100 K higher at 373.15 K. The constant of proportionality between fundamental and Kelvin temperatures is the Boltzmann constant, i.e.

$$\tau = kT \tag{5}$$

where $k = 1.38066 \times 10^{-23} JK^{-1}$. By a quantum statistical analysis, based on the Schrödinger equation, of N particles in a box in thermal equilibrium at temperature T, one can then show that the mean energy per translational degree of freedom of a free particle is $\tau/2$ so the total energy of the particles is $\frac{3}{2}N\tau = \frac{3}{2}NkT$ (see [4], p. 72).

Given the quantum statistical definition of τ , and the definition of T in terms of τ and the triple point of water, one could, in principle, compute k in terms of the atomic constants such as e, m_e, and h if one could solve the Schrödinger equation for water at its triple point in all its terrible complexity. But that is a hopeless task, so one must turn to empirical determinations of the proportionality constant based on experiments that link the macroscopic and microscopic aspects of the world.

A link between the microscopic and macroscopic was reported by Johnson in 1928 [1] in a paper paired in the Physical Review with one by H. Nyquist [2] that provided a rigorous theoretical explanation based on the principles of classical thermal physics. Johnson had demonstrated experimentally that the mean square of the voltage across a conductor is proportional to the resistance and absolute temperature of the conductor and does not depend on any other chemical or physical property of the conductor. At first thought, one might expect that the magnitude of Johnson noise must depend in some way on the number and nature of the charge carriers. In fact Nyquist's theory involves neither e nor N. It yields a result in agreement with Johnson noise observations and a formula for the mean square of the noise voltage which relates the value of the Boltzmann constant to quantities that can be readily measured by electronic methods and thermometry.

NYQUIST'S THEORY OF JOHNSON NOISE

Two fundamental principles of thermal physics are used:

1. The second law of thermodynamics, which implies that between two bodies in thermal equilibrium at

the same temperature, in contact with one another but isolated from outside influences, there can be no net flow of heat;

2. The equipartition theorem of statistical mechanics [4], which can be stated as follows:

> "Whenever the hamiltonian of a system is homogeneous of degree 2 in a canonical momentum component, the thermal average kinetic energy associated with that momentum is $kT/2$, where T is the Kelvin temperature and k is Boltzmann's constant. Further, if the hamiltonian is homogeneous of degree 2 in a position coordinate component, the thermal average potential energy associated with that coordinate will also be $kT/2$."

   If the system includes the electromagnetic field, then the Hamiltonian includes the term $(E^2 + B^2)/8\pi$ in which E and B are canonical variables corresponding to the q and p of a harmonic oscillator for which (with p and q in appropriate units) the hamiltonian, is $(q^2 + p^2)/2$.

Nyquist's original presentation of his theory is magnificient, please see the Junior Lab e-library for a copy.

Nyquist invoked the second law of thermodynamics to replace the apparently intractable problem of adding up the average thermal energies in the modes of the electromagnetic field around a conductor of arbitrary shape and composition with an equivalent problem of adding up the average thermal energies of the readily enumerated modes of electrical oscillation of a transmission line shorted at both ends. Each mode is a degree of freedom of the dynamical system consisting of the electromagnetic field constrained by the boundary conditions imposed by the transmission line. According to the equipartition theorem, the average energy of each mode is kT, half electric and half magnetic.

The Nyquist formula for the differential contribution dV_j^2 to the mean square voltage across a resistor of resistance R in the frequency interval df due to the fluctuating emfs corresponding to the energies of the modes in that interval is

$$dV_j^2 = 4RkTdf \tag{6}$$

To measure this quantity, or rather its integral over the frequency range of the pass band in the experiment, one must connect the resistor to the device by means of cables that have a certain capacitance C which shunts (short circuits) a portion of the signal, thereby reducing its RMS voltage. The equivalent circuit is shown in Figure 1. The differential contribution $dV_j'^2$ to the signal

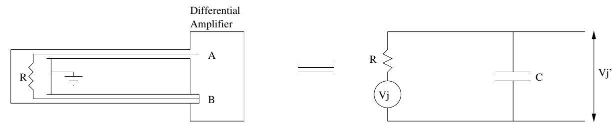

FIG. 1: Equivalent circuit of the thermal emf across a conductor of resistance R connected to a measuring device with cables having a capacitance C

presented to the input of the measuring device (in our case the A–input of a low-noise differential preamplifier) is a fluctuating voltage with a mean square value

$$dV_j'^2 = 4R_f kTdf \tag{7}$$

where

$$R_f = \frac{R}{1 + (2\pi fCR)^2} \tag{8}$$

This equation is equivalent to Equation (A9) in Appendix B and results from AC circuit theory.

Attention was drawn earlier to an analogy between the mean square of the Johnson noise voltage across a conductor and the pressure of a gas on the walls of a container. Both are proportional to kT and the number of degrees of freedom of the system. The big difference between the two situations is that the number of translational degrees of freedom per mole of gas is the "unknown" quantity $3N$, while the number of oscillation modes within a specified frequency interval in the transmission line invoked by Nyquist in his theory is readily calculated from the laws of classical electromagnetism.

Since the Boltzmann constant is related to the number of accessible quantum states, one might well ask:

> Where is Planck's constant, h, which fixes the actual number of accessible states?

The answer is that the Nyquist theorem in its original form, like the classical Rayleigh-Jeans formula for the spectral distribution of blackbody radiation, is valid only in the range of frequencies where $hf << kT$, in other words, at frequencies sufficiently low that the minimum excitation energy of the oscillations is small compared to kT. At 300K and 100 kHz, $kT = 4 \times 10^{-14}$ ergs (0.04 eV) and $hf = 6 \times 10^{-22}$ ergs. Thus at room temperature kT is $\sim 10^8$ times the minimum energy of an oscillation mode with a frequency near 100 kHz. The exact quantum expression for the mean energy ϵ of each oscillation mode, noted by Nyquist in his paper, is

$$\epsilon = \frac{hf}{e^{hf/kT} - 1} \tag{9}$$

which reduces to $\epsilon = kT$ for $hf << kT$ over the range of frequencies and temperatures encountered in this experiment. On the other hand, for the electrical noise at the input of a radio telescope operated at liquid helium temperature for measurement of the $\sim 2.7K$ cosmic background radiation at frequencies near the peak of the black body spectrum, i.e. around 10^{11} Hz.

EXPERIMENT

In the present experiment you will actually measure an amplified version of a portion of the Johnson noise power spectrum. The portion is defined by the "pass band" of the measurement chain which is determined by a combination of the gain characteristics of the amplifier and the transmission characteristics of the low-pass/high-pass filters that are included in the measurement chain to provide an adjustable and sharp control of the pass band. The combined effects of amplification and filtration on any given input signal can be described by a function of frequency called the effective gain and defined by

$$g(f) = \left(\frac{V_0(f)}{V_i(f)} \right) \tag{10}$$

where the right side is the ratio of the RMS voltage V_0 out of the bandpass filter to the RMS voltage V_i of a pure sinusoidal signal of frequency f fed into the amplifier. **A critical task in the present experiment is to measure the square of the effective gain as a function of frequency for the particular settings of the measurement chain which are used in the subsequent measurement of Johnson noise.**

When the input of the measurement chain is connected across the conductor under study, the contribution dV_J^2 to the total mean square voltage out of the bandpass filter in a differential frequency interval is

$$dV_J^2 = [g(f)]^2 \; dV_j'^2 \tag{11}$$

where $V_j'^2$ is the voltage measured across the resistor. We obtain an expression for the measured total mean square voltage by integrating Equation 11 over the range of frequencies of the pass band. Thus

$$V_J^2 = 4RkTG \tag{12}$$

where the quantity G is defined by

$$G \equiv \int_0^\infty \frac{[g(f)]^2}{1 + (2\pi f CR)^2} df \tag{13}$$

The rationale behind this integration is that over any given time interval t the meandering function of time that is the instantaneous noise voltage across the resistor can be represented as a Fourier series consisting of a sum of sinusoids with discrete frequencies $n/2t$, n=1, 2, 3 ...., each with a mean square amplitude equal to the value specified by the equipartition theorem. When the Fourier series is squared the cross terms are products of sinusoids with different frequencies, and their average values are zero. Thus the expectation value of the squared voltage is the sum of the expectation values of the squared amplitudes, and in the limit of closely spaced frequencies as $t \to \infty$, the sum can be replaced by an integral.

Given the linear dependence of V_J^2 on T in Eq. 13, it is evident that one can use the Johnson noise in a resistor as a thermometer to measure absolute temperatures. A temperature scale must be calibrated against two phenomena that occur at definite and convenient temperatures such as the boiling and melting points of water, which fix the centigrade scale at 100° C and 0°C, respectively. In the present experiment you will take the centigrade calibrations of the laboratory thermometers for granted, and determine the centigrade temperature of absolute zero as the zero-noise intercept on the negative temperature axis.

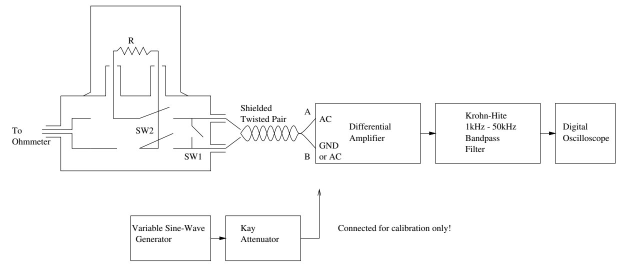

FIG. 2: Block diagram of the electronic apparatus for measuring Johnson noise.

PROCEDURE OVERVIEW

The experiment consists of the following parts:

1. Calibration of the measurement chain;
2. Measurement of $V_J'^2 = V_R^2 - V_S^2$, where V_R = RMS voltage at the output of the band-pass filter with the resistor in place; and V_S = RMS voltage with the resistor shorted); for various resistors and temperatures;
3. From the data, determine the Boltzmann constant;
4. Determine the centigrade temperature of absolute zero.

Figure 2 is a schematic diagram of the apparatus showing the resistor R mounted on the terminals of the aluminum box, shielded from electrical interference by an inverted metal beaker, and connected through switches

SW1 and SW2 to the measurement chain or the ohmmeter. The measurement chain consists of a low-noise differential amplifier, a band-pass filter, and a digital oscilloscope. Sinusoidal calibration signals are provided by a function generator.

To minimize the problem of electrical interference in the measurement of the low-level noise signals it is essential that all cables be as short as possible. The two cables that connect the resistor to the 'A' and 'B' input connectors of the differential amplifier should be tightly twisted, as shown, to reduce the flux linkage of stray AC magnetic fields.

The digital oscilloscope emits a variable magnetic field from its beam-control coil which will have a devastating effect on your measurements unless you keep it far away ($\geq$ 5 feet) from the noise source. Take special care to avoid this problem when you arrange the components of the measurement chain on the bench.

There are various possible choices of settings and procedure for carrying out measurements that will yield a value of k. The following recipe works satisfactorily, but you may well devise a better one. The procedure is based on the assumption that the component of V_R not generated in the resistor can be measured with sufficient accuracy by

1. opening switch SW2,
2. unplugging the connections to the ohmmeter and temperature meter, and
3. shorting the resistor with switch SW1.

The pass band of the Krohn-Hite filter is fixed with the 3dB High Pass frequency = 1 KHz and the Low Pass 3dB point = 50 KHz.

DETAILED PROCEDURE

Calibrate the measurement chain

Typical RMS voltages of the Johnson noise signals you will measure are of the order of several microvolts. The digital oscilloscope can measure the RMS voltage of both periodic and random signals over a dynamic range of somewhat more than 10^3, from several millivolts to several volts. Thus, with the differential amplifier set to a nominal gain of 1000, the microvolt noise signals will be amplified sufficiently to be measured in the millivolt range of the oscilloscope. To determine the overall amplification of the amplifier/filter combination, one can feed a sinusoidal test signal with an RMS voltage V_i in the millivolt range to the 'A' AC-Coupled of the PAR amplifier (with the 'B' input grounded), and measure the RMS voltage V_0 of the output of the filter in the volt range of the oscilloscope. The gain of the system at the frequency of the test signal is $g(f) = Vo/Vi$.

Measuure the variation of the test signal RMS voltage and the gain of the measurement chain as a function of frequency

Select the sinusoid signal of the function generator (FG) and adjust its amplitude so the RMS voltage V_i out of the FG is $\sim$ 2V as measured on the digital oscilloscope on Channel 1. Use a tee to simultaneously send the test signal through the Kay attenuator with 60 dB (1000) of attenuation to the 'A' input of the amplifier with the 'B' input grounded. Display this attenuated signal on Channel 2 of the oscilloscope.

Without touching the amplitude control of the FG, measure and record the RMS voltage of the FG signal at several frequencies over the range that will pass the filter ($\sim$0.5 kHz to $\sim$100 kHz). Plot the measured RMS voltages against the frequencies so as to have a handy database of the input RMS voltages required for the determination of $g(f)$. You should expect to find that the RMS voltage varies only slightly over most of the frequency range of interest. Also Measure the RMS voltages out of the filter at the same frequencies to define accurately the curve of the square of the gain $[g(f)]^2$ versus f. Plot g^2 against f as you go along to check the consistency and adequacy of your data.

The digital oscilloscope can be set up for these measurements as follows:

1. Press '1' under the channel '1' input.
2. Select 'COUPLING:AC'; 'Bandwidth limit: ON'.
3. Select 'VOLTAGE Measurement: Vrms'.
4. Select 'Frequency Measurement'.
5. Press 'DISPLAY', and select 'AV 8' for test sinusoid measurements, and 'normal' for noise measurements. (Note: You must select 'normal' for noise measurements since the RMS voltage of the average of n random wave forms approaches zero as $n \rightarrow \infty$. On the other hand, the RMS voltage of the average of many wave forms consisting of a constant sinusoid plus random noise approaches the RMS voltage of the pure sinusoid.)
6. Adjust the digital scope amplitude and sweep-speed controls so that several ($\sim$5-10) cycles of the sinusoid appear on the screen.

To reduce errors of measurement and obtain an error assessment, you can use the 'stop' and 'run' buttons to advantage. Press 'run' and then 'stop', record the V_{rms} and frequency displayed at the bottom of the screen; repeat n times (e.g., n=5) at each setting. For each setting compute the mean V_{rms}, and the standard error of the mean ($=\frac{\sigma}{\sqrt{n-1}}$).

Measure ${V'}_J^2$ for a variety of resistors

With a ∼20 KΩ resistor and the amplifier gain set to 1000, the RMS voltage at the output of the bandpass filter should be in the range of several millivolts. About half the RMS voltage is noise generated in the amplifier itself. Interference pickup may vary. Since all the contributions to the measured RMS voltage are statistically uncorrelated, **they add in quadrature**. To achieve accurate results it is essential to make repeated measurements with each resistor with the shorting switch across the conductor alternately opened and closed. The measure of the mean square Johnson noise is

$$V_J'^2 = V_R^2 - V_S^2 \tag{14}$$

where V_R and V_S are the RMS voltages measured with the shorting switch open and closed, respectively.

Measure the Johnson noise at room temperature in ∼10 metal film and/or wire-wound resistors with values from 10^3 to 10^6 ohms. Mount the resistors in the alligator clips projecting from the aluminum test box equipped with a single pole single throw (SPST) shorting switch, a double pole double throw (DPDT) routing switch, and connections for a thermistor for use in the later temperature measurement. Cover the resistor and its mounts with a metal beaker to shield the input of the system from electrical interference. After each noise measurement measure the resistance of the resistor: plug a digital multimeter into the pin jacks on the aluminum box and flip the DPDT switch on the sample holder to the resistance measuring position. Before each noise measurement, be sure to disconnect the multimeter (to avoid introducing extraneous electrical noise) and flip the DPDT switch back to the noise-measurement position.

Plot $V_J'^2/R$ against R . See how precisely you can account for the results by a proper choice for the values of k and C. According to equation 12 the value of k can be expressed in terms of measured quantities and G, which is a function of R and C:

$$k = \frac{V_J'^2}{4RTG} \tag{15}$$

The factor G must be recalculated by numerical integration for each new trial value of R and C. In principle, if you use the correct value of C in your calculations of G, then the values of k obtained from Equation 15 should cluster around a mean value close to the value for $R \to 0$ and should not vary systematically with R. Using the best values of k and C derived in this way, you can plot a calculated curve of $V_J'^2/R$ against R for comparison with your data.

Question: What happens as $R \to \infty$?

Measure Johnson noise as a function of temperature

Measure the Johnson noise in a metal film or wire wound resistor over a range of temperatures from that of liquid nitrogen (77 K) to ∼ 150°C. Clip the ∼ 30kΩ resistor with the thermistor attached to the aligator clips, and plug the thermistor leads into the color-coded sockets, taking care with the delicate wires of the thermister. To make a temperature measurement, plug the thermometer into the color-coded socket on the side of the box. The high temperatures are obtained by inverting the box with the mounted test resistor and thermister and placing it in the cylindrical oven which contains heater tape and insulation. Connect the heater to the Variac and set the Variac at ∼40 V. Measure the resistance, the RMS voltages of the Johnson noise and the background as the temperature rises. For cold measurements, fill the Dewar flask with liquid nitrogen and place it in the large metal beaker. Then rest the inverted box on the lip of the metal beaker to assure good electrical shielding.

According to the Nyquist theory the points representing the measured values of $V_J'^2/4RG$ plotted against T (°C degrees) should fall on a straight line with a slope equal to the Boltzmann constant, and an intercept on the temperature axis at the centigrade temperature of absolute zero. Note that if the resistance of the conductor varies significantly with temperature, then G must be evaluated separately at each temperature, i.e. the integral of Equation 13 must be evaluated for each significantly different value of the resistance.

1. Make a plot of $V_J'^2/4RG$ against T (in °C degrees).
2. Derive a value and error estimate of k from the slope of the temperature curve.
3. Derive a value and error estimate of the centigrade temperature of absolute zero.

SHOT NOISE

A current source in which the passage of each charge carrier is a statistically independent event (rather than a steady flow of many charge carriers) necessarily delivers a "noisy" current, i.e., a current that fluctuates about an average value. Fluctuations of this kind are called "shot noise". The magnitude of such fluctuations depends on the magnitude of the charges on the individual carriers. Thus a measurement of the fluctuations should, in principle, yield a measure of the magnitude of the charges.

Consider a circuit consisting of a battery, a capacitor in the form of a photodiode, a resistor, and an inductor, connected in series as illustrated in Figure 3a. Illumination of the photodiode with an incoherent light source causes electrons to be ejected from the negative electrode

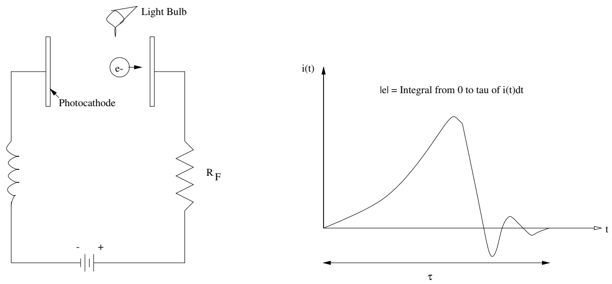

FIG. 3: (a) Schematic diagram of a circuit in which the current consists of a random sequence of pulses generated by the passage of photoelectrons between the electrodes of a photodiode. (b) Schematic representation of the current pulse due to the passage of one photoelectron.

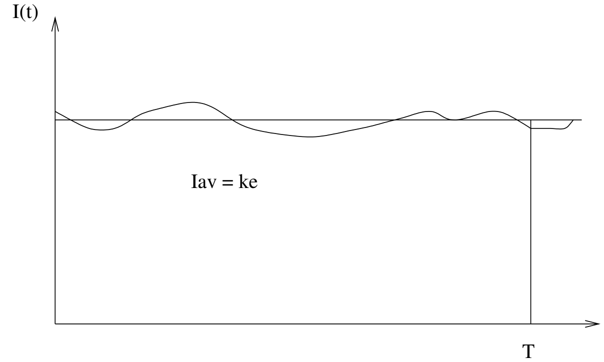

FIG. 4: Plot of a fluctuating current against time with a straight line indicating the long-term average current.

in a random sequence of events. Each ejected photoelectron, carrying a charge of magnitude e, is accelerated to the positive electrode, and during its passage between the electrodes it induces an increasing current in the circuit, as shown in Figure 3b.

When the electron hits the positive plate the current continues briefly due to the inductance of the circuit and a damped oscillation ensues. The shape of the current pulse depends on the initial position, speed and direction of the photoelectron as well as the electrical characteristics of the circuit. The integral under the curve is evidently the charge e. If the illumination is strong enough so that many events occur during the duration of any single electron pulse, then the current will appear as in Figure 4, in which the instantaneous current $I(t)$ fluctuates about the long-term average current I_{av} .

In this experiment you will measure, as a function of I_{av}, the mean square voltage of the output of an amplifier and a band pass filter system whose input is the continuously fluctuating voltage across R_F . According to the theory of shot noise a plot of this quantity against the average current should be a straight line with a slope proportional to e . The problem is to figure out what the proportionality factor is.

THEORY OF SHOT NOISE

The integral under the curve of current versus time for any given pulse due to one photoelectron event is e, the charge of the electron. If the illumination is constant and the rate of photoelectric events is very large, then the resulting current will be a superposition of many such waveforms $i_k(t)$ initiated at random times T_k with a "long term" average rate we will call K, resulting in a fluctuating current with an average value $I_{av} = Ke$, as illustrated in Figure 4. The fluctuating component of such a current was called "shot noise" by Schottky in 1919 who likened it to the acoustic noise generated by a hail of shot striking a target.

The fluctuating current is

$$I(t) = \Sigma_k i_k(t) \tag{16}$$

and its mean square during the time interval T is

$$< I^2 >= \frac{1}{T}\int_0^T [\Sigma_k i_k(t)]^2 dt \tag{17}$$

The problem is to derive the relation between the measurable properties of the fluctuating current and e. Although the derivation is somewhat complicated (see Appendix C), the result is remarkably simple: within the frequency range $0 < f << \frac{I_{av}}{e}$ the differential contribution to the mean square of the total fluctuating current from fluctuations in the frequency interval from f to $f + df$ is (see Appendix B)

$$d < I^2 >= 2eI_{av}df \tag{18}$$

Suppose the fluctuating current flows in a resistor of resistance R_F connected across the input of an amplifier-filter combination which has a frequency-dependent gain $g(f)$. During the time interval T the voltage developed across the resistor, IR_F, can be represented as a sum of Fourier components with frequencies $m/2T$, where $m = 1, 2, 3, ...$, plus the zero frequency (DC) component of amplitude $I_{av}R_F$. Each component emerges from the amplifier-filter with an amplitude determined by the gain of the system for that frequency. The mean square of the sum of Fourier components is the sum of the mean squares of the components (because the means of the cross terms are all zero). Thus, in the practical limit of a Fourier sum over closely spaced frequencies, we can express the mean square voltage of the fluctuating output signal from the amplifier-filter as the integral

$$V_0^2 = 2eI_{av}R_F^2 \int_0 [g(f)]^2 df + V_A^2 \tag{19}$$

where V_A^2 has been added to represent the constant contribution's of the amplifier noise , and Johnson Noise in R_f to the total mean square voltage, and where the DC term is omitted because the DC gain of the amplifier-filter is zero.

To get a feel for the plausibility of the shot noise formula one can imagine that the current in the photodiode circuit is a step function representing the amount of charge ne released in each successive equal time interval of duration τ divided by τ, i.e., the mean current in each interval $\frac{ne}{\tau}$. The expectation value of n is $< n >= K\tau$. We assume there is no statistical correlation between the numbers of events in different intervals. According to Poisson statistics the variance of n (mean square deviation from the mean) is the mean, i.e.,

$$< (n- < n >) >^2 = < n^2 > - < n >^2 = < n >= K\tau \quad (20)$$

It follows that the mean square value of the current over time would be

$$\begin{aligned} < I^2 > &= < \left(\frac{ne}{\tau}\right)^2 > \\ &= \left(\frac{e}{\tau}\right)^2 < n^2 > \\ &= \left(\frac{e}{\tau}\right)^2 \left[< n > + < n >^2 \right] \\ &= \frac{I_{av}e}{\tau} + I_{ave}^2 \quad (21) \end{aligned}$$

which shows that the fluctuating term is proportional to eI_{av} as in the exact expression for the differential contribution, Equation 18.

Actually the current at any given instant from an illuminated photodiode is the sum of the currents due to the photoelectrons ejected during the previous brief time interval. Thus the currents at any two instants separated in time by less than the duration of the individual pulses are not statistically independent. Moreover, the simple scheme provides no handle on the frequency spectrum of the noise which one must take into account in evaluating the response of the measurement chain. One approach to a rigorous solution is presented in Appendix B. Others are possible.

SHOT NOISE EXPERIMENTAL PROCEDURE

The procedure has three parts:

1. Calibration of the gain of the measurement chain as a function of frequency;
2. Measurement of the mean square noise voltage at the output of the measurement chain as a function of the average current in the diode circuit as it is varied by changing the intensity of illumination.
3. Calculation of the charge of the photoelectrons.

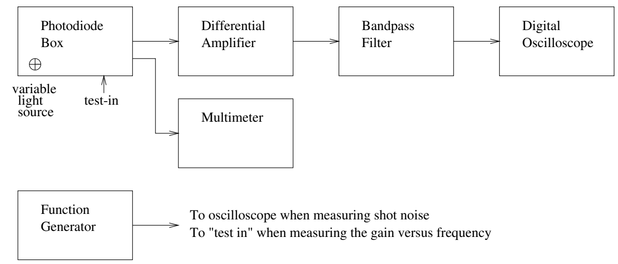

FIG. 5: Block diagram of the experimental arrangement for measuring shot noise.

CALIBRATION OF THE MEASUREMENT CHAIN

Figure 5 is a block diagram of the electronic apparatus, and Figure 6 is a diagram of the diode circuit and preamplifier. The current $I(t)$ in the photodiode circuit is converted to a voltage $V = IR_F$ at the point indicated in Figure 6 by the operational amplifier with precision feedback resistors in the first stage of the preamplifier inside the photodiode box. This voltage is filtered to remove frequencies <100 Hz and is fed to the second stage where it is amplified by a factor of ~10. The output signal is further amplified with the differential amplifier, filtered with the bandpass filter and measured with the RMS voltmeter.

The photodiode box has a test input for calibration of the overall gain of the measurement chain as a function of the frequency. The typical shot noise RMS voltage across the precision resistor R_F is of the order of 10 microvolts. Since the gain of the circuit in the photodiode box is ~10, a gain of 100 in the amplifier will yield a total gain of 10^3 and bring the signal in the pass band of the filter up to the ~10 millivolt level that can be readily measured by the digital oscilloscope. As in the Johnson noise calibration, you can determine the effective gain of the amplifier-filter system as a function of frequency by feeding a millivolt sinusoid signal of measured RMS voltage from the function generator into the test input of the phototube box and measuring the RMS voltage of the signal out of the filter. You can use the same band pass settings of the filter as in the Johnson noise measurement.

Calibration

Select the sine wave output of the function generator, set the the RMS voltage to ~2 mV, and feed the signal directly to the digital oscilloscope. Without touching the amplitude control on the function generator, measure the RMS voltage for several frequencies over the range of the

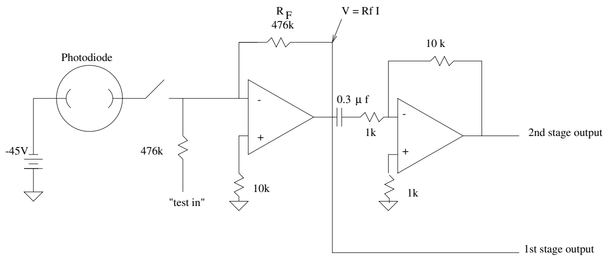

FIG. 6: Diagram of the photodiode and preamplifier circuit.

filter band pass and plot the result so you have a handy data base for the subsequent gain measurements.

Measure the gain of the measurement chain as a function of frequency. Make sure the photodiode tube voltage is switched off so that the tube is an open circuit with no photoelectric current. Switch on the amplifier voltage on the phototube box. Feed test signals of various frequencies into the 'test input' and measure the RMS voltage at the output of the bandpass filter. Use the signal averaging feature and the 'stop', 'run' buttons on the oscilloscope to enhance the accuracy of your measurements. Plot the values of g^2 as you go along to assess where you need more or less data to define accurately the gain squared integral.

MEASUREMENT OF THE AVERAGE CURRENT AND THE CURRENT NOISE

Remove the cable from the test input and cover the input plug with the cap provided to minimize the pickup of electrical noise from the outside. Set the multi-meter to "volts" on the 200 mV scale and plug it into the "first stage output" to measure the voltage $R_F I$. Leave the rest of the measurement chain just as it was when you calibrated it. Verify that the flashlight bulb is actually working by opening the box and looking. If it doesn't glow brightly when the voltage is turned up, fix it.

Record the RMS voltage and the multimeter reading for various settings of the light bulb voltage control. Many repeated measurements at each light intensity will beat down the random errors.

ANALYSIS

Plot V_o^2 as a function of the combined quantity

$$2R_F^2 I_{av} \int_0^{\infty} g^2(f) df \tag{22}$$

From the slope of this line determine the charge on the electron.

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SUGGESTED THEORETICAL TOPICS

- The Nyquist theorem.
- Shot noise theory

APPENDICES

A MECHANICAL EXPERIMENT TO DETERMINE k

Before turning to a detailed consideration of the Johnson noise experiment, it is amusing to consider the possibility of a mechanical determination of k with a macroscopic system having one degree of freedom, namely a delicate torsion pendulum suspended in a room in thermal equilibrium (i.e. no drafts, etc.) at temperature T. The degree of freedom is the angular position θ with which is associated the potential energy $\frac{1}{2}\kappa\theta^2$. According to the equipartition theorem (see below), the mean thermal potential energy is

$$\frac{1}{2}\kappa < \theta^2 >= \frac{1}{2}kT \tag{23}$$

where κ is the torsion constant of the suspension, and $< \theta^2 >$ is the mean square angular displacement of the pendulum from the equilibrium orientation. Thus, in principle, by measuring $< \theta^2 >$ over a time long compared to the period, one can determine k . To judge what this might require in practice, imagine a torsion balance consisting of a tiny mirror (for reflecting a laser beam) suspended by a 0.5 mil tungsten wire 10 feet long. Such a suspension has a torsion constant of the order of 10^{-3} dyne cm rad^{-1}. According to Equation 23, at 300 K the value of $< \theta^2 >^{1/2}$, i.e. the RMS value of the angular deflection, would be about 1 arc second. Such an experiment might be possible, but would be exceedingly difficult.

DERIVATION OF THE RMS THERMAL VOLTAGE AT THE TERMINALS OF AN RC CIRCUIT

Figure 1 shows the circuit equivalent to the resistor and coaxial cables that are connected to the PAR preamplifier for the measurement of Johnson noise. The equivalent circuit consists of a voltage source of the fluctuating thermal emf V_J in series with an ideal noiseless resistor of resistance R and a capacitor of capacitance C. According to Faraday's Law, the integral of the electric field around the RC loop is zero, so

$$V_J = IR + \frac{Q}{C} \tag{24}$$

According to charge conservation (from Ampere's Law and Gauss' Law), the current into the capacitor equals the rate of change of the charge on the capacitor, so

$$I = \frac{dQ}{dt} \tag{25}$$

We seek an expression in terms of $d < V_J^2 >$, R, and C for the contribution to the RMS voltage across the terminals in a narrow frequency range, i.e.

$$d < V'^2_J >= d < Q^2 > /C \tag{26}$$

Consider one Fourier component of the fluctuating thermal emf across the resistor, and represent it by the real part of $\nu_J = \nu_0 \exp^j \omega t$, where $j = \sqrt{-1}$. The resulting current is the real part of $i = i_0 \exp^j \omega t$, the charge on the capacitor is the real part of its integral $q = -(j/\omega)i$, and the desired output voltage is the real part of $\frac{q}{C} = -(\frac{j}{\omega C} i)$. Substituting the expressions for i and q into Equation 24 and canceling the time-dependent terms, we find

$$\nu_0 = (R - \frac{j}{\omega C})i_0 \tag{27}$$

Solving for i_0 we obtain the relation

$$i_o = \frac{\nu_0}{R - \frac{j}{\omega C}} \tag{28}$$

so

$$\nu'_J = \frac{q}{C} = -(\frac{j}{\omega C})i = \frac{-j\nu_J}{\omega RC - j} \tag{29}$$

The statistically independent contribution which this mode gives to the measured total mean square noise voltage is the square of its amplitude which we find by multiplying ν'_J by its complex conjugate:

$$< \nu'^2_J >= \frac{< \nu_J^2 >}{1 + (\omega RC)^2} \tag{30}$$

Summing all such contributions in the differential frequency range df, we obtain

$$< dV'^2_J >= 4R_f kT df \tag{31}$$

where

$$R_f = \frac{R}{1 + (2\pi fRC)^2} \tag{32}$$

DERIVATION OF THE SHOT NOISE EQUATION

Following is an abbreviated version of the shot noise theory given by Goldman (1948). We begin by expressing the current in the photodiode circuit due to a single event that occurs at time T_k, like that depicted in Figure 3b, as a Fourier series over a long time interval from 0 to T. Calling this current pulse $i(t-T_k)$ we represent it as a Fourier series:

$$i(t-T_k)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi nt}{T}+b_n\sin\frac{2\pi nt}{T}\right) \quad (33)$$

where

$$a_0=\frac{2}{T}\times\int_0^T i(t-T_k)dt=\frac{2e}{T} \quad (34)$$

$$a_n=\frac{2}{T}\times\int_0^T i(t-T_k)\cos\frac{2\pi nt}{T}dt=\frac{2e}{T}\cos\frac{2\pi nT_k}{T} \quad (35)$$

$$b_n=\frac{2}{T}\times\int_0^T i(t-T_k)\sin\frac{2\pi nt}{T}dt=\frac{2e}{T}\sin\frac{2\pi nT_k}{T} \quad (36)$$

with $i(t-T_k)=e\delta(t-T_k)$.

The area under the curve in Figure 3b is the charge e of one electron; it represents the "impulse" of the shot, and is accounted for in the Fourier representation by the lead term in the series whose coefficient is given by Equation 34. To justify Equations 35 and 36 in the context of the present experiment we note that the gain of the amplifier-filter system used in this measurement is different from zero only for frequencies such that $f=n/T<<1/\tau$. Consequently we can confine our calculation of the Fourier coefficients to those for which $n\tau<<T$. It follows that the cos and sin factors in the integrands of equations 35 and 36 do not vary significantly over the range of t in which $i(t-T_k)$ differs from 0, and that they can therefore be taken outside their integrals with their arguments evaluated at the instant of the event. In other words, the function representing the current impulse of a single event acts like a delta-function. Substituting the expressions for a_0, a_n, and b_n from equations 34, 35, and 36 into 33 we obtain

$$i(t-T_k)=\frac{e}{T}+\frac{2e}{T}\sum_{n=1}^{\infty}\cos\left[\frac{2\pi n(t-T_k)}{T}\right] \quad (37)$$

We suppose now that many such events pile up to produce the total current at any given instant. We seek a way to add the currents due to the individual events to obtain the differential contribution $d<I_0^2>$ in the frequency interval df to the mean square of the sum. We use the well known fact that the mean square of the sum of all the Fourier components is the sum of the mean squares of the individual components (the mean values of the cross-frequency terms in the squared Fourier series are all zero). The quantity $d<I_0^2>$ is therefore the sum of the mean squares of the individual contributions in the frequency range df. To evaluate it we first focus attention on the n^{th} Fourier component which we represent by

$$c_n\cos\left(\frac{2\pi nt}{T}-\phi_n\right) \quad (38)$$

to which the k^{th} event contributes the quantity

$$\frac{2e}{T}\cos\left(\frac{2\pi n(t-T_k)}{T}\right) \quad (39)$$

The mean square value of the n^{th} component is c_n^2; our immediate problem is to evaluate the quantity c_n^2. Since the events occur at random times from 0 to T, their contributions to the n^{th} component have random phases which are distributed uniformly from 0 to 2π. Consequently, we must add them as vectors. To do this we first group them according to their phase. The expected number with phases between ϕ and $\phi+d\phi$ is $\frac{d\phi}{2\pi}KT$.

$$q=\frac{d\phi}{2\pi}KT \quad (40)$$

Combining this with equations 38 and 39 we find that the average value of the sum of the contributions with phase angles in the range from ϕ to $\phi+d\phi$ for the Fourier component of frequency n/T is

$$\frac{d\phi}{2\pi}KT\frac{2e}{T}\cos\left(\frac{2\pi nt}{T}-\phi\right)\frac{Ke}{\pi}\cos\left(\frac{2\pi nt}{T}-\phi\right) \quad (41)$$

We now represent each one of the preliminary sums given by equation 40 by a differential vector in a two dimensional phase diagram. Added head to tail in order of increasing phase, these vectors form a closed circular polygon of many sides. If instead of the exact expected numbers of events contributing to each differential vector we use the actual numbers q_1, q_2, ..., then in general the polygon will not quite close due to statistical fluctuations in these numbers. The line segment that closes the gap is the overall vector sum for this particular Fourier component. It will have a random direction and a tiny random length that represents the net effect of the fluctuations. Its x and y components will have zero expectation values, but finite variances (mean square values). Each contribution to the total x component is a number of events that obeys a Poisson distribution with a variance equal to its expectation value. By a well known theorem of statistics (used frequently in error analysis) the variance of the sum is the sum of variances. Thus the mean square of the x component of the vector representation of the n^{th} Fourier component is

$$\begin{aligned}\frac{4e^2}{T^2}(q_1\cos^2\phi_1 + q_2\cos^2\phi_2 + ...) &= \\ &= \frac{4e^2}{T^2}\frac{KT}{2\pi}\int_0^{2\pi}\cos^2\phi d\phi \\ &= \frac{2e^2K}{T} \qquad (42)\end{aligned}$$

The mean square of the y components has the same value. And since the vector itself is the hypotenuse of the right triangle formed by the x and y components, the mean square of its length is the sum of the mean squares of the two components. Thus

$$< c_n^2 >= \frac{4e^2K}{T} \qquad (43)$$

and it follows that the contribution of the n^{th} Fourier component to $d < I_0^2 >$ is $\frac{<C_n^2>}{2} = \frac{2e^2K}{T}$. The frequency of the n^{th} component is $f = n/T$, so the number of Fourier components corresponding to a frequency bandwidth df is $dn = Tdf$. Therefore the contribution to the mean square value of the sum of the Fourier components in the frequency range df is

$$d < I_0^2 >= \frac{2e^2K}{T}Tdf = 2e^2Kdf \qquad (44)$$

The average current due to the many events is $I_{av} = Ke$. Thus the final expression for the differential contribution to the mean square of the fluctuating component of the current from the differential frequency interval df is just that given by Equation 18 from which follows Equation 19, the desired formula for the relation between measured quantities and e.