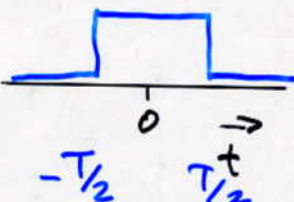
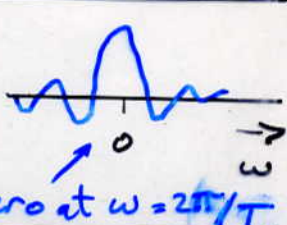
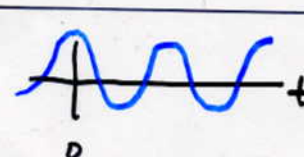
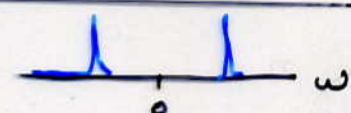
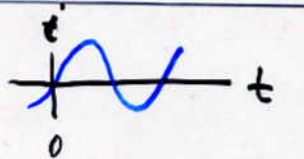
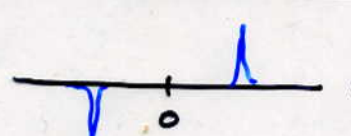
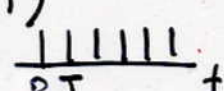
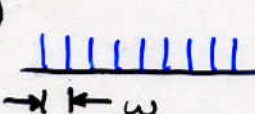


## Math of spectral response

Fourier Transform  $F(t) = \tilde{G}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} G(\omega) d\omega$

$$G(\omega) = \tilde{F}(t) = \int_{-\infty}^{\infty} e^{-i\omega t} F(t) dt$$

F-T Pairs :

F(t)	G(ω)
	$\int_{-T/2}^{T/2} e^{-i\omega t} dt = \frac{2 \sin \frac{\omega T}{2}}{\omega}$ "sinc" function zero at $\omega = 2\pi/T$ 
$\frac{1}{2\pi} e^{i\omega_0 t}$	$\delta(\omega - \omega_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(\omega - \omega_0)t} dt$ Dirac δ function
$\frac{1}{\pi} \cos \omega_0 t$ 	$\delta(\omega - \omega_0) + \delta(\omega + \omega_0)$ Even 
$\frac{i}{\pi} \sin \omega_0 t$ 	$\delta(\omega - \omega_0) - \delta(\omega + \omega_0)$ Odd 
$\llbracket \rrbracket_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$  <p>Dirac Comb</p>	$\frac{\omega_0}{2\pi} \llbracket \rrbracket_{\omega_0}(\omega) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$  <p>Area under each peak = <math>\omega_0 / 2\pi</math></p>

# Convolution

$$A(\omega) * B(\omega) = \int_{-\infty}^{\infty} A(\omega - \omega') B(\omega') d\omega' = B(\omega) * A(\omega)$$

Prove by substitution  $\uparrow$   
 $\omega' \rightarrow \omega - \omega'' \quad d\omega' \rightarrow -d\omega''$

## Convolution Theorem

$$\widetilde{A(t) \cdot B(t)} = \widetilde{A(t)} * \widetilde{B(t)}$$

product convolution

### Example 1



Q: What is FT of triangle function?

### Example 2

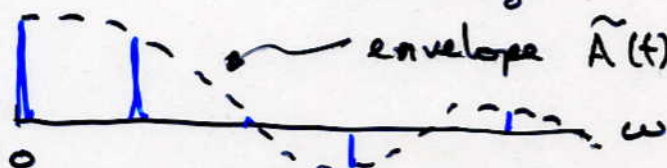


### Example 3 FT of symmetric square wave

step 1: from theorem

$$\text{Square Wave}(t) = \text{Dirac}(t) * \text{Triangle}(t) = \text{Dirac}(t) * A(t)$$

FT is product  $\frac{\omega_0}{2\pi} \text{Dirac}(\omega) \cdot \widetilde{A(t)}$



odd harmonics missing

# Spectral density

Fluctuating quantity  $X(t)$

assume stationary, and  $\bar{X} = 0$

$$\text{Variance of } X = \overline{(X - \bar{X})^2} = \overline{X^2}$$

Define "spectral density"  $S_x(\omega)$

$S_x(f) \Delta f \equiv$  mean square Fourier component  
in interval  $\Delta f$  @ frequency  $f$

Since Fourier components at different frequencies  
are independent, noise powers add.

Consider noise  $X(t)$   $0 < t < T$

$$\text{expand: } x(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega_0 t} \quad \omega_0 = \frac{2\pi}{T}$$

$$\text{where } a_n = \frac{1}{T} \int_0^T x(t) e^{-in\omega_0 t} dt, \quad a_{-n} = a_n^*$$

Fourier component of  $X(t)$  at frequency  $n\omega_0$  is

$$x_n = a_n e^{in\omega_0 t} + a_n^* e^{-in\omega_0 t}$$

Mean square Fourier component (ensemble average):

$$\overline{x_n^2} = 2 \overline{a_n a_n^*} = S_x(f) \Delta f \quad \Delta f \equiv \frac{1}{T}$$

↳ Other terms vanish due to random phase

Powerful theorem:

$$\text{Consider } 2 \overline{a_n a_n^*} = \frac{2}{T^2} \int_0^T \int_0^T X(u) X(v) e^{i n \omega_0 (v-u)} du dv$$

[integral over area in  $u-v$  plane]

can be transformed to

$$\frac{2}{T} \int_{-\infty}^{\infty} X(u) X(u+s) e^{i n \omega_0 s} ds$$

Note that  $\langle X(u) X(u+s) \rangle$  is autocorrelation of  $X$   
 $\uparrow$   
ensemble average

$S_x(f)$  is twice the FT of  $\overline{X(u) X(u+s)}$  !

Correlation function  $K(s) \equiv \langle X(t) X(t+s) \rangle$

$K(0) = \text{dispersion of } X \quad (\bar{X} = 0)$

$$K(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega s} d\omega$$

$$S(\omega) = \int_0^{\infty} K(s) e^{-i\omega s} ds$$

Symmetry:  $K^*(s) = K(s)$  and  $K(-s) = K(s)$   
 $S^*(\omega) = S(\omega)$   $S(-\omega) = S(\omega)$

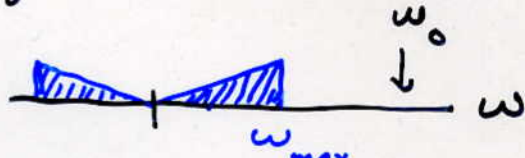
$$\Rightarrow \langle X^2 \rangle = K(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) d\omega = \frac{1}{2\pi} \int_0^{\infty} S_+(\omega) d\omega$$
$$S_+(\omega) = 2S(\omega)$$

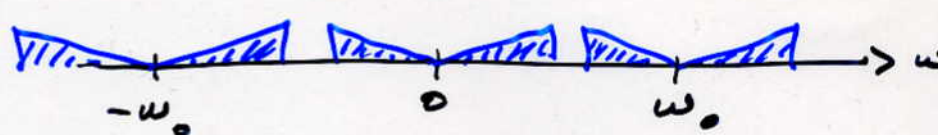
Wiener - Khintchine

# Sampling Theorem of Information theory

Sampling a time series  $V(t)$  is a multiplication by a Dirac comb with interval  $T$ :  $V(t) \cdot \sum_T \delta(t)$

In frequency domain:  $\tilde{V}(t) * \sum_{\omega_0} \delta(\omega)$

Given a spectrum  $\tilde{V}(t) = V(\omega)$    $\omega_0$   
 $\omega_{max}$

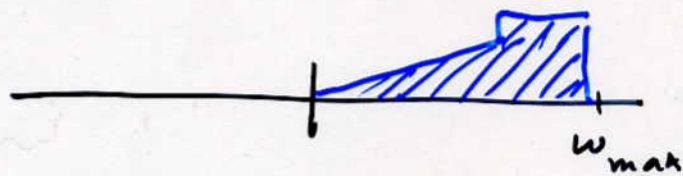
Convolution gives   $\omega$   
 $-\omega_0$   $0$   $\omega_0$

when  $\omega_{max} \leq \omega_0/2$ , no ambiguity.

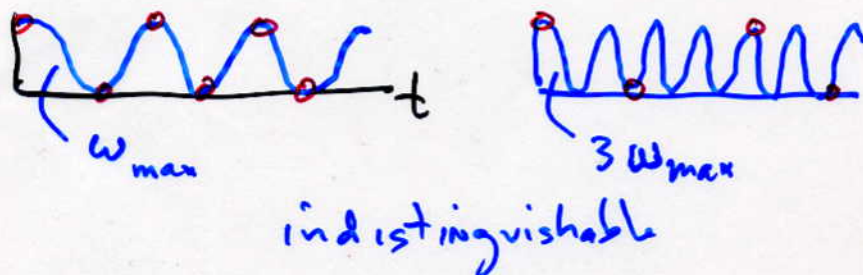
If  $\omega_{max} > \omega_0/2$  spectrum is aliased.

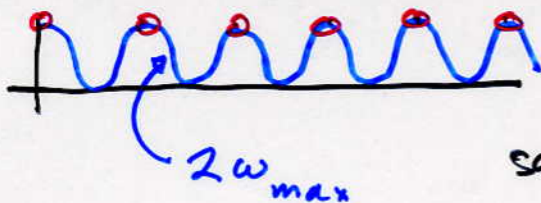


recovered spectrum  $0 - \omega_{max}$  will be indistinguishable from



Time stream illustration:





sample looks like  $\omega = 0$  !

Therefore, spectrum must be cut off at  $\omega_{max}$  and be sampled at  $\omega_{smp} > 2\omega_{max}$  to avoid aliasing.

Examples include Brillouin zone theory in solids, FT spectroscopy, speech/music sampling, etc.

## Digital Spectrum Analyzer

- samples time stream  $V(t)$  over duration  $T$  at interval  $\tau$
- computes sine and cosine Fourier components
- computes sum of squares to obtain  $S_V(f)$
- averages derived  $S_V(f)$  over several intervals  $T$

Independent Fourier components are squared separately so noise at one frequency does not mix with noise at another frequency.

Features:

- Good way to measure coherent signal superimposed on random noise — with no reference.
- Out of phase noise included, so S/N is  $\sqrt{2}$  worse.
- Resolution  $\Delta f = 1/T$
- Sampling rate must be  $> 2$  times highest analysis freq.
- Available systems not well configured for long time averages or signal amplitude measurement.

2  
1

# Fluctuation - Dissipation

Consider a thermodynamic system in equilibrium.

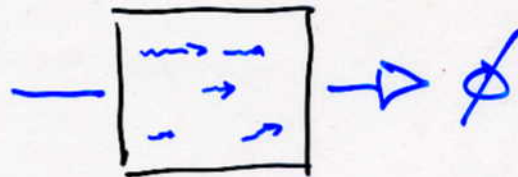
microscopic relaxation  
time  $\tau^*$



$\frac{1}{2}kT$  / degree of freedom

Fluctuation spectrum.

Consider non-equilibrium response on  
timescales  $\tau \gg \tau^*$



Macroscopic response (decay from non-equilibrium) is related to equilibrium fluctuations.

Macroscopic dissipation  $\leftrightarrow$  Equilibrium fluctuations

- Brownian motion
- Electrical thermal noise

Fundamental

