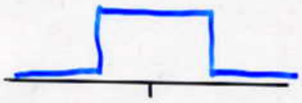
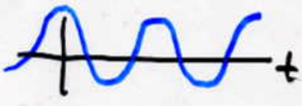
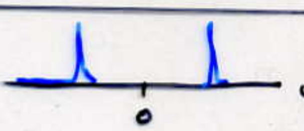
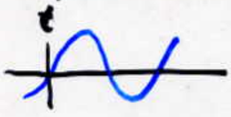
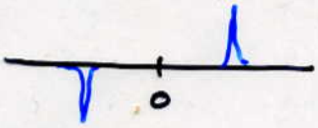
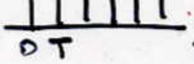
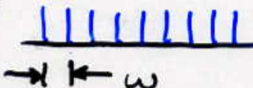
Math of spectral response

Fourier Transform $F(t) = \tilde{G}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\, G(\omega)\, d\omega$

$$G(\omega) = \tilde{F}(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\, F(t)\, dt$$

F-T Pairs:

$F(t)$	$G(\omega)$
	$\int_{-T/2}^{T/2} e^{-i\omega t} dt = \frac{2}{\omega}\sin\frac{\omega T}{2}$  "sinc" function zero at $\omega = 2\pi/T$
$\frac{1}{2\pi} e^{i\omega_0 t}$	$\delta(\omega-\omega_0) = \frac{1}{2\pi}\int^{\infty} e^{-i(\omega-\omega_0)t} dt$ Dirac δ function
$\frac{1}{\pi}\cos\omega_0 t$ 	$\delta(\omega-\omega_0)+\delta(\omega+\omega_0)$ Even 
$\frac{i}{\pi}\sin\omega_0 t$ 	$\delta(\omega-\omega_0)-\delta(\omega+\omega_0)$ Odd 
$Ш_T(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$  Dirac Comb	$\frac{\omega_0}{2\pi} Ш_{\omega_0}(\omega) = \frac{\omega_0}{2\pi}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$  Area under each peak $= \omega_0/2\pi$

Convolution

$$A(\omega) * B(\omega) = \int_{-\infty}^{\infty} A(\omega-\omega')\, B(\omega')\, d\omega' = B(\omega) * A(\omega)$$

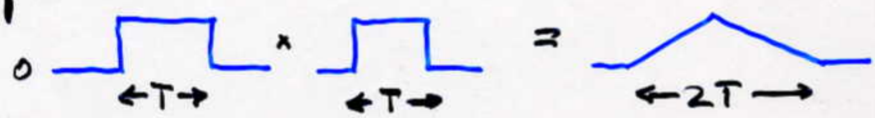
Prove by substitution

$\omega' \to \omega - \omega''$ $\quad d\omega' \to -d\omega''$

Convolution Theorem $\quad \widetilde{A(t)\cdot B(t)} = \widetilde{A(t)} * \widetilde{B(t)}$

product $\qquad$ convolution

Example 1

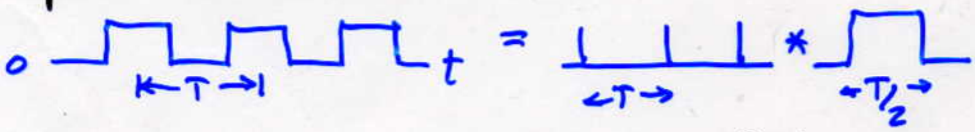


Q: What is FT of triangle function?

Example 2

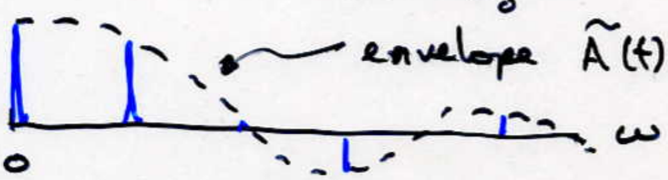
Example 3 FT of symmetric square wave

step 1: from theorem



$= Ш_T(t) * A(t)$

FT is product $\frac{\omega_0}{2\pi} Ш_{\omega_0}(\omega) \cdot \widetilde{A(t)}$



odd harmonics missing

Spectral density

Fluctuating quantity $x(t)$

assume stationary, and $\bar{x} = 0$

Variance of $x = \overline{(x-\bar{x})^2} = \overline{x^2}$

Define "spectral density" $S_x(\omega)$

$S_x(f)\,\Delta f \equiv$ mean square Fourier component in interval Δf @ frequency f

Since Fourier components at different frequencies are independent, noise powers add.

Consider noise $x(t)$ $\quad 0 < t < T$

expand: $\quad x(t) = \sum_{n=-\infty}^{\infty} a_n e^{in\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T}$

where $\quad a_n = \frac{1}{T}\int_0^T x(t) e^{-in\omega_0 t}\,dt \quad , \quad a_{-n} = a_n^*$

Fourier component of $x(t)$ at frequency $n\omega_0$ is

$$x_n = a_n e^{in\omega_0 t} + a_n^* e^{-in\omega_0 t}$$

Mean square Fourier component (ensemble average):

$$\overline{x_n^2} = 2\,\overline{a_n a_n^*} = S_x(f)\,\Delta f \qquad \Delta f \equiv \frac{1}{T}$$

Other terms vanish due to random phase

Powerful theorem:

Consider $2\overline{a_n a_n^*} = \frac{2}{T^2}\int_0^T\int_0^T X(u)X(v)e^{in\omega_0(v-u)}\,du\,dv$

[integral over area in u-v plane]

can be transformed to

$$\frac{2}{T}\int_{-\infty}^{\infty} X(u)\,X(u+s)\,e^{in\omega_0 s}\,ds$$

Note that $\langle X(u)\,X(u+s)\rangle$ is autocorrelation of X

ensemble average

$S_x(f)$ is twice the FT of $\overline{X(u)\,X(u+s)}$!

Correlation function $K(s) \equiv \langle x(t)\,x(t+s)\rangle$

$K(0)$ = dispersion of X $\quad (\bar{x}=0)$

$$K(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,e^{i\omega s}\,d\omega$$

$$S(\omega) = \int_0^{\infty} K(s)\,e^{-i\omega s}\,ds$$

symmetry: $K^*(s) = K(s)$ and $K(-s) = K(s)$

$S^*(\omega) = S(\omega) \qquad S(-\omega) = S(\omega)$

$\Rightarrow \langle x^2\rangle = K(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega = \frac{1}{2\pi}\int_0^{\infty} S_+(\omega)\,d\omega$

$S_+(\omega) = 2S(\omega)$

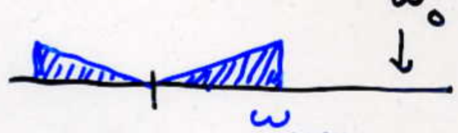
Wiener - Khintchine

Sampling Theorem of Information theory

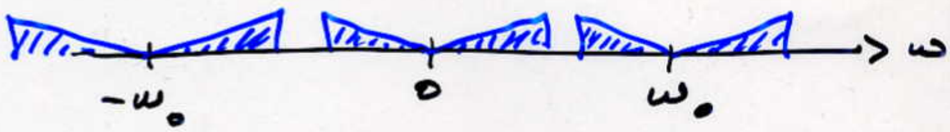
Sampling a time series $V(t)$ is a multiplication by a Dirac comb with interval τ: $V(t) \cdot Ш_{\tau}(t)$

In frequency domain: $\tilde{V}(t) * Ш_{\omega_0}(\omega)$

Given a spectrum $\tilde{V}(t) = V(\omega)$



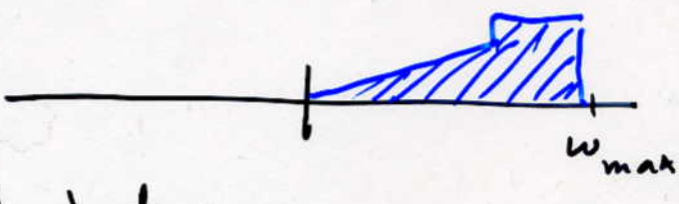
Convolution gives



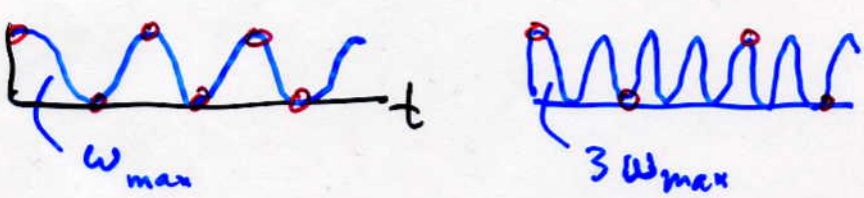
when $\omega_{max} \leq \omega_0/2$, no ambiguity.

If $\omega_{max} > \omega_0/2$ spectrum is _aliased_.

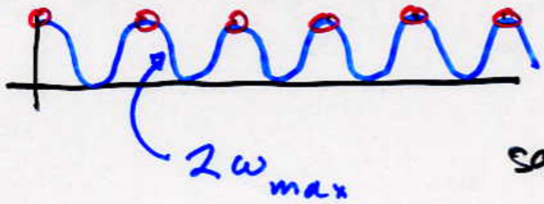
recovered spectrum $0 - \omega_{max}$ will be indistinguishable from



Time stream illustration:



indistinguishable



sample looks like $\omega = 0$!

Therefore, spectrum must be cut off at ω_{max} and be sampled at $\omega_{samp} > 2\omega_{max}$ to avoid aliasing.

Examples include Brillouin zone theory in solids, FT spectroscopy, speech/music sampling, etc.

Digital Spectrum Analyzer

- samples time stream $V(t)$ over duration T at interval τ
- computes sine and cosine Fourier components
- computes sum of squares to obtain $S_v(f)$
- averages derived $S_v(f)$ over several intervals T

Independent Fourier components are squared separately so noise at one frequency does not mix with noise at another frequency.

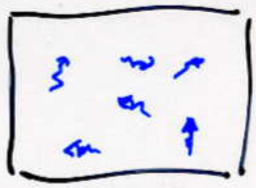
Features:

- ☐ Good way to measure coherent signal superimposed on random noise — with no reference.
- ☐ Out of phase noise included, so S/N is $\sqrt{2}$ worse.
- ☐ Resolution $\Delta f = 1/T$
- ☐ Sampling rate must be > 2 times highest analysis freq.
- ☐ Available systems not well configured for long time averages or signal amplitude measurement.

Fluctuation - Dissipation

Consider a thermodynamic system in equilibrium.

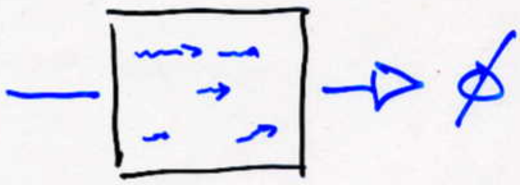
microscopic relaxation time τ^*



$\frac{1}{2}kT$ / degree of freedom

Fluctuation spectrum.

Consider non-equilibrium response on timescales $\tau \gg \tau^*$



Macroscopic response (decay from non-equilibrium) is related to equilibrium fluctuations.

Macroscopic dissipation $\Leftrightarrow$ Equilibrium fluctuations

- Brownian motion
- Electrical thermal noise

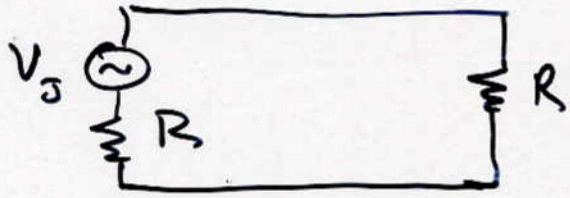
Fundamental

Example: Johnson noise

Single mode Black Body radiation in classical limit $h\nu \ll kT$

Power/mode $P_\nu \, d\nu = kT \, d\nu$

Consider transmission line with matched resistors $R = Z_0$ at temperature T radiating power $kT\,d\nu$ at each other:



Noise current $I_{rms} = V_{rms} / 2R$

Power dissipated in resistor on right

$= kT\,d\nu = I^2 R$

fluctuation

dissipation

$$\therefore \quad V_{rms} = (4kTR\,\Delta f)^{1/2}$$

Dissipation $\to$ Fluctuation

Fluctuation $\to$ Dissipation