Simple Circuits

Resistor R $I \to$ current $R \equiv$ resistance

V voltage   Ohm's Law: $V = IR$

Capacitor C $I \to$ Charge $Q = CV \Rightarrow I = C\frac{dV}{dt}$

V

Inductor L $I \to$ $V = L\frac{dI}{dt}$

V

Impedance + Reactance

Voltages and currents as complex numbers:

$$V(t) = V_0 \cos(\omega t + \phi) \equiv \mathrm{Re}\left[e^{i\omega t} V_0 e^{i\phi} = (a+ib)e^{i\omega t} \right]$$

Reactance: $\quad I(t) = \mathrm{Re}\left[V_0 e^{i\omega t} / (-i/\omega C) \right] = \mathrm{Re}\, \frac{V_0 e^{i\omega t}}{X_C}$

X $\quad X_C = -\frac{i}{\omega C}$

$X_L = i\omega L$

Impedance (generalized Ohm's Law):

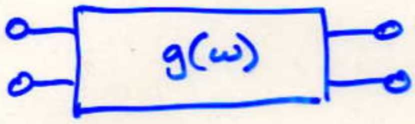
Z $\quad V = IZ$

Series elements: $Z = \sum_n Z_n$

Parallel elements: $Z^{-1} = \sum_n Z_n^{-1}$

$Z_R = R \qquad Z_C = -\frac{i}{\omega C} \qquad Z_L = i\omega L$

Transfer function of a network

$X(\omega)$ 

$Y(\omega) = g(\omega)\, X(\omega)$

$g(\omega)$ is complex : $g_0 e^{i\phi}$ $\qquad \phi$ = phase shift

If $X(\omega) = X_0 e^{i\omega t}$, $\quad Y(\omega) = X_0 g_0 e^{i(\omega t + \phi)}$

In time domain $X(t) = \frac{1}{2\pi} \int X(\omega) e^{-i\omega t} d\omega$

$$Y(t) = \frac{1}{2\pi} \int g(\omega)\, X(\omega)\, e^{-i\omega t}$$

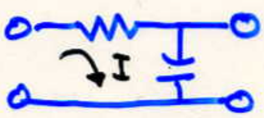
$$Y(t) = \widetilde{g(\omega)} * \widetilde{X(\omega)} = g(t) \cdot X(t)$$

where $g(t) = \frac{1}{2\pi} \int e^{-i\omega t} g(\omega)\, d\omega$

Example 1: To measure $g(t)$ apply pulse $X(t) = \delta(t)$
$\rightarrow$ ouput $Y(t) = g(t)$

Example 2: For broadband amplifier $g(\omega)$ = constant.
$Y(t) \sim X(t)$ $\rightarrow$ no distortion

Example 3: Chose single pole Low-pass filter for $g(\omega)$

V_i 

V_o $\qquad V_i = V_{oi} e^{i\omega t}$

$$I = V_i / Z = V_i / \left(\frac{1}{i\omega C} + R \right)$$

$$V_o = I / i\omega C$$

$$g = \frac{V_o}{V_i} = \frac{1}{1 + i\omega RC} = \frac{1 - i\omega RC}{1 + \omega^2 R^2 C^2}$$

note $\tan\phi = \omega RC$

time constant $\tau = RC$

$$|g| = \left|\frac{V_o}{V_i}\right| = \frac{1}{(1+\omega^2R^2C^2)^{1/2}}$$

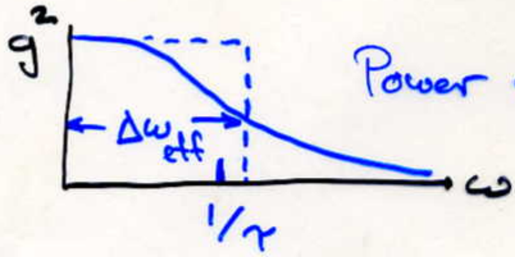
For $\omega\tau >> 1$ $\quad |g| \sim \omega^{-1}$ $\quad$ Power $\sim \omega^{-2}$

For each factor of 2 in frequency [octave] power decreases $\times \frac{1}{4}$

Define $dB = 10 \log_{10}$

$\therefore \; 10 \log_{10}(\frac{1}{4}) \approx$ 6 dB / Octave "rolloff"

Effective Bandwidth

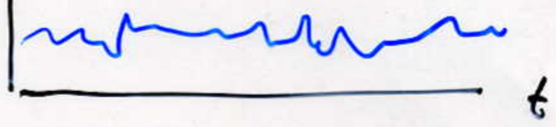


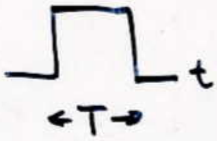
$$\Delta\omega_{eff} = \frac{\int_0^\infty |g(\omega)|^2 d\omega}{|g(0)|^2}$$

$$= \frac{1}{\tau}\int_0^\infty \frac{d(\omega\tau)}{1 + \omega^2\tau^2} = \frac{\pi}{2\tau}$$

$$\Rightarrow \Delta f_{eff} = \frac{1}{4\tau} = \frac{1}{4RC}$$

Example 4: Running average
(usually implemented digitally)

Given time stream $x(t)$ 

convolve with  $\equiv A(t)$

$$\widehat{x(t) * A(t)} \equiv g(\omega) \cdot X(\omega) \quad \text{where}$$

gain $g(\omega) = \frac{2}{\omega T} \sin\left(\frac{\omega T}{2}\right)$ sinc function

then $\Delta\omega_{eff} = \frac{2}{T} \int_0^\infty \left(\frac{2}{\omega T}\right)^2 \sin^2\left(\frac{\omega T}{2}\right) d\left(\frac{\omega T}{2}\right) = \frac{\pi}{T}$

$\therefore \quad \Delta f_{eff} = \frac{1}{2T}$

Useful properties of Power Spectrum

Variance of $x(t) = \langle (x-\bar{x})^2 \rangle = \overline{x^2} = \int_0^\infty S_x(f)\,df$

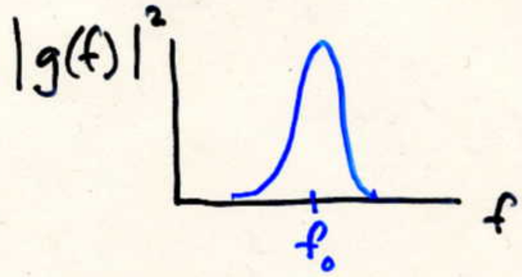
(for stationary process), $\bar{x} = 0$

$\overline{x^2}$ = area under $S_x(f)$

Apply a noise signal $x(t)$ with power spectrum $S_x(f)$ to a circuit with complex gain $g(f)$.

Variance of output is $\overline{Y^2} = \int_0^\infty S_x(f)\,|g(f)|^2\,df$

In special case where $|g(f)|^2$ has a relatively narrow peak compared with structure in $S_x(f)$



$$\overline{Y^2} = S_x(f_0)\int_0^\infty |g(f)|^2\,df = S_x(f_0)\,|g(f_0)|^2\,\Delta f_{eff}$$

This is the basis of effective bandwidth

$$B_{eff} = \Delta f_{eff} = \int_0^\infty \frac{|g(f)|^2}{|g(f_0)|^2}\,df$$

$\Rightarrow$ In practice, a narrow band filter can be used to measure $S_x(f)$

Power spectrum