PHY 123/253 Statistics and Error Propagation

Types of errors, parent and sample distributions. Error propagation.

1. READING

- Bevington & Robinson Chapters 1-4.2
- Helpful: J. R. Taylor "An Introduction to Error Analysis"

2. SUMMARY/OVERVIEW:

When quoting your results in PHY 123/253, you are expectred to give a solid statistical error and an estimate of the systematic error. Note: statistical errors come solely from REPEATED, independent, measurements.

Errors are not mistakes but uncertainties in measurements:

a) Random Errors, σ - jitter of measurements around the true value, μ.

b) Systematic Errors, Δ - deviation from truth by faulty knowledge/equipment.

If we make N measurements $x_1, x_2, \dots x_N$ and quote the result

$$x_{result} = x_{best} \pm s_x \pm \Delta x \tag{1}$$

then usually:

$$x_{best} = \langle x \rangle = \frac{1}{N} \sum x_i \qquad \text{mean} \tag{2}$$

$$s_x = \frac{1}{N-1} \sum (x_i - \langle x \rangle)^2 \qquad \text{std. dev.} \tag{3}$$

$$\Delta x = \text{estimate of unmeasured `systematic' effects} \tag{4}$$

If x_i came from a parent or population distribution with probability density $p(x)$, the population mean $\mu = \lim_{N\to\infty} \langle x \rangle$ and variance $\sigma_x^2 = \lim_{N\to\infty} s_x^2$.

Note: $\langle x^2 - \langle x \rangle^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$.

Some common parent distributions are:

a) Gaussian:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \text{with } \sigma = \sqrt{\mathrm{N}} \text{ for counting experiments} \tag{5}$$

b) Poisson:

$$p(x) = \frac{\mu^x}{x!} e^{-\mu} \qquad \text{with variance } \sigma = \sqrt{\mu}. \tag{6}$$

c) Lorentzian:

$$p(x) = \frac{1}{\pi} \frac{\Gamma/2}{(x-\mu)^2 + (\Gamma/2)^2} \qquad \text{FWHM} : |x - \mu| = \pm\Gamma/2 \tag{7}$$

In general, these distributions govern experiments with: a) high statistics ($\mu \geq 20$), b) low statistics ($\mu < 20$) and c) distributions of photons with line width $\Gamma = \hbar/E$.

3. ERROR ANALYSIS

Counting Experiments:
Result $= (N \pm \sqrt{N})$ for distributions (5) and (6).

Continuous Experiments:
Result $= T \pm \sigma_T$ (temperature T_i, voltage, etc...) T_i are most likely Gaussian distributed, if your measurements are independent (i.e. the measurements are uncorrelated and do not depend on each other). The variance σ you obtain from fitting a Gaussian to your distribution of values depends, for example, on the coarseness of the scale of your thermometer, etc.

3.1. Error Propagation

You determine the height x of a building by letting a stone drop and measuring the time t with a watch.

$$x = \frac{1}{2} g t^2 \quad \longrightarrow \quad x = \langle x \rangle \pm \sigma_x$$

From your watch accuracy, σ_t, you want to know the error in x, σ_x. Then in this example:

$$\frac{\sigma_x}{x} \simeq \frac{\sigma_t}{x} \left(\frac{\partial x}{\partial t} \right) = \frac{\sigma_t}{x} g t = 2 \frac{\sigma_t}{t}.$$

In general, if we evaluate $x(w)$ from a measured w with σ_w, then

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$$\sigma_x^2 \simeq \sigma_w^2 \left(\frac{\partial x}{\partial w}\right)^2 \tag{8}$$

and for more parameters $w_1, w_2, \ldots w_m$:

$$\sigma_x = \sqrt{\sum_{i=1}^{m} \left(\sigma_w^i \frac{\partial x}{\partial w_i}\right)^2}. \tag{9}$$

Example, if $x = w_1/w_2$, (or $w_1 \cdot w_2$)

$$\frac{\sigma_x}{x} = \sqrt{\frac{\sigma_{w1}^2}{w_1^2} + \frac{\sigma_{w2}^2}{w_2^2}},$$

i.e. fractional errors add in quadrature.

Having made N measurements we quote our best (maximum likelihood) values as:

Distribution Mean

$$\langle x \rangle = \frac{1}{N} \sum x_i \approx \mu \qquad = \frac{\sum (x_i/\sigma_i)^2}{\sum (1/\sigma_i)^2} \tag{10}$$

Distribution Uncertainty

$$\sigma_x = \frac{\sigma_{\text{one measurement}}}{\sqrt{N}} \qquad = \sqrt{\frac{N}{N-1}} \sqrt{\frac{\sum (x_i - \langle x \rangle)^2/\sigma_i^2}{\sum (1/\sigma_i)}} \tag{11}$$

where the first value is for N measurements of equal error and the second is for the case of combining measurements of different σ_i by error weighting.

4. PROBLEMS

1. Bevington & Robinson (2003) exercise 2.15
2. Bevington & Robinson (2003) exercise 2.16
3. Bevington & Robinson (2003) exercise 4.5