Experiments in special relativity using Compton scattering of gamma rays

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Some simple undergraduate laboratory experiments are described, which verify the energy–momentum relationship of special relativity. These experiments have been designed either to be used as classroom demonstrations or to be carried out by second-year students.

INTRODUCTION

An experiment is described which demonstrates large, controllable, and unambiguous relativistic effects using readily available, inexpensive nuclear instrumentation. The data can be acquired in a single laboratory session or, as a demonstration, in a single lecture period. The experimental design is focused on proving that the mean energy of γ rays scattered at angle θ is independent of the scattering material and establishing the relationship between the energy shift and θ. The analysis of the data emphasizes the fact that the scattering electron can be treated classically in x-ray scattering whereas when energetic γ rays are scattered its relativistic mass change is clearly evident. The underlying ideas of conservation of energy and momentum are the same as those demonstrated in experiments in classical mechanics using air tables and pucks.

Although the consistency of special relativity with experiment is established as thoroughly as that of any scientific theory, few persuasive tests can be demonstrated in undergraduate laboratories. Consequently a pedagogical tradition has been established of presenting the subject as a logical deduction from cleverly contrived thought experiments. This approach is not unique—the uncertainty principle is sometimes presented in a similar way—but it is unusual and some students find it particularly difficult to accept. We hope that the present experiments will contribute to the development of an alternative presentation of the subject in a more conventional form.

The phenomenon, commonly called the Compton effect, which we use to demonstrate relativity has a most complex history which has been presented recently both in great detail[1] and in abbreviated form.[2,3] That history reveals that even the earliest measurements of γ-ray scattering indicated consistently that there was a large angle-dependent energy loss. The effects were so large as to be persuasive despite the need to rely on crude measurements of absorption coefficients.[4,5] The data which generated a decade of debate came from measurements[8] of x-ray scattering in which the effects were small enough to cause confusion even when measured by crystal diffraction spectrometers of high resolution.

The analysis of the early experiments seems to have been done either classically or using both quantum and relativity concepts. The possibility of emphasizing the relativity aspects by using the photon concept with either classical or relativistic dynamics for the electron has received little attention.

THEORETICAL REMARKS

There are many ways in which the results of a Compton experiment can be used and we discuss below three analyses which we believe are particularly good pedagogically. In all cases we assume scattering by a free electron initially at rest and that the γ/x rays can be treated as quanta with an energy momentum relation $E = pc$. Energy and momentum conservation in the collision process are also assumed. We describe below how knowledge of the incident photon energy and a measurement of the scattered photon energy as a function of scattering angle can be used to (1) compare nonrelativistic and relativistic Compton formulas, (2) determine the energy–momentum relation of the electron, or (3) measure the rest-mass energy of the electron.

Our choice of assumptions versus deductions in the Compton experiment is somewhat arbitrary and, for example, students may find our acceptance of $E = pc$ on the basis of the "classical" Maxwell equations and the rejection of the "classical" Newtonian expression $K = (1/2)m_0v^2$ artificial. In response we simply admit that any one experiment can have only a limited set of objectives. Students could also point out that the scattering process does not involve free electrons, but here at least the free-electron assumption can be motivated experimentally by the complete insensitivity of the results to the scattering material.

Our notation is as follows. An incident photon of momentum $\mathbf{p}_i$ and energy $E_i = p_ic$ is scattered through an angle θ. The final photon has momentum $\mathbf{p}_s$ and energy $E_s = p_sc$ and the electron, initially at rest, has picked up the difference momentum $\mathbf{p} = \mathbf{p}_i - \mathbf{p}_s$ and energy $K = E_i - E_s$. These two conservation equations lead to

$$p^2c^2 - K^2 = 2E_iE_s(1 - \cos\theta). \qquad (1)$$

In Newtonian or nonrelativistic (NR) mechanics the energy momentum relation of the electron is

$$\text{(NR)} \qquad p^2 = 2m_0K \qquad (2a)$$

whereas the correct relativistic (R) expression is

$$\text{(R)} \qquad p^2 = 2m_0K + K^2/c^2. \qquad (2b)$$

If we eliminate the momentum p in Eq. (1) using the expressions (2), and in addition eliminate K in favor of the explicitly measured $E_i - E_s$, we obtain either

$$\text{(NR)} \qquad m_0c^2\left(\frac{1}{E_s} - \frac{1}{E_i}\right) + \cos\theta - \frac{(E_i - E_s)^2}{2E_iE_s} = 1 \qquad (3a)$$

or

$$\text{(R)} \qquad m_0c^2\left(\frac{1}{E_s} - \frac{1}{E_i}\right) + \cos\theta = 1 \qquad (3b)$$

as expressions which can be directly tested experimentally. A useful exercise for students is to tabulate the three ex-

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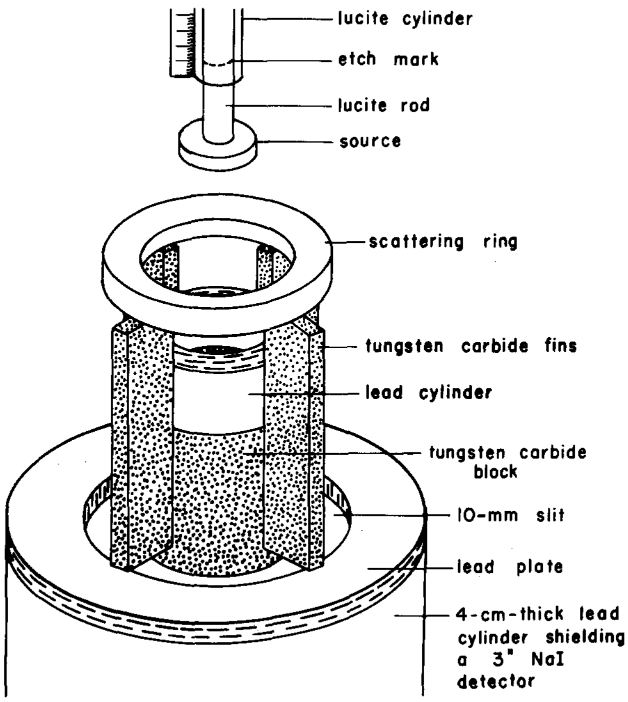

Fig. 1. Arrangement of source, scattering ring and shielding about 3 × 3 in. NaI(Tl) detector. The tungsten carbide components are standard shapes supplied by the manufacturer and are held together by silicone calking compound.

pressions in the left-hand side of Eq. (3a) in addition to their sum and the sum of the first two only. For low E_i or small angles cancellations in the first term lead to uncertainties which can mask the effect of the third term which in magnitude is always less than $(1/2)[E_i(1-\cos\theta)/m_0c^2]^2$. For the experimental arrangement described here, this sets a few 100 keV as a practical lower limit on the energy of the photons below which the relativistic and nonrelativistic formulas cannot be distinguished.

Another approach to the Compton experiment is obtained by adopting the point of view that it is a scattering experiment from which can be deduced the electron energy–momentum relation. Equation (1) together with the relation $K = E_i - E_s$ completely specifies p^2c^2 in terms of experimentally determined quantities and hence as a function of K. A particularly striking way of displaying the data is to plot p^2c^2/K, which from Eq. (1) is

$$\frac{p^2c^2}{K} = \frac{2E_iE_s(1-\cos\theta)}{E_i - E_s} + E_i - E_s \qquad (4)$$

vs K. The theoretical prediction for this quantity as given by Eqs. (2) is a straight line with either *zero* or *unit* slope.

Finally, if the relativistic energy–momentum relation for the electron is accepted, one can deduce the rest mass of the electron from Eq. (2b) rewritten as

$$\text{(R)} \qquad m_0c^2 = \frac{1}{2}\left(\frac{p^2c^2}{K} - K\right) = \frac{E_iE_s(1-\cos\theta)}{E_i - E_s} \qquad (5)$$

and the relativistic mass $m = m_0 + K/c^2$ which may be written

$$\text{(R)} \qquad mc^2 = m_0c^2 + (E_i - E_s). \qquad (6)$$

Here again one must guard against loss of accuracy through the cancellation of large quantities and this sets the same practical lower limit of a few 100 keV on the photon energies.

EXPERIMENTAL ARRANGEMENT

The basic measurement is simply the determination of the energy spectrum of γ radiation which reaches a detector after scattering at a known angle. A conventional scintillation spectrometer with a 3 × 3 in. NaI(Tl) detector, a linear amplifier, and a multichannel analyser (MCA) was used. We omit electronic details since any commercial amplifier or MCA produced in the last decade would have adequate specifications for the system.

The important experimental details are the arrangements of shielding and scattering materials as indicated in Fig. 1. The system was designed to be compatible with a 50-μCi source of ^{137}Cs (contained in a plastic disc), giving an acceptable signal-to-background ratio as seen in Fig. 2. The choice of a point source with a ring-shaped scatterer and entrance aperture gives a large improvement in efficiency over a conventional point-to-point-to-point scattering arrangement. Efficiency is achieved at obvious cost in definition of the scattering angle; the angular resolution full wave at half maximum (FWHM) is $\Delta\theta \sim 10°$. The width of the peak in the Compton energy spectrum will be approximately

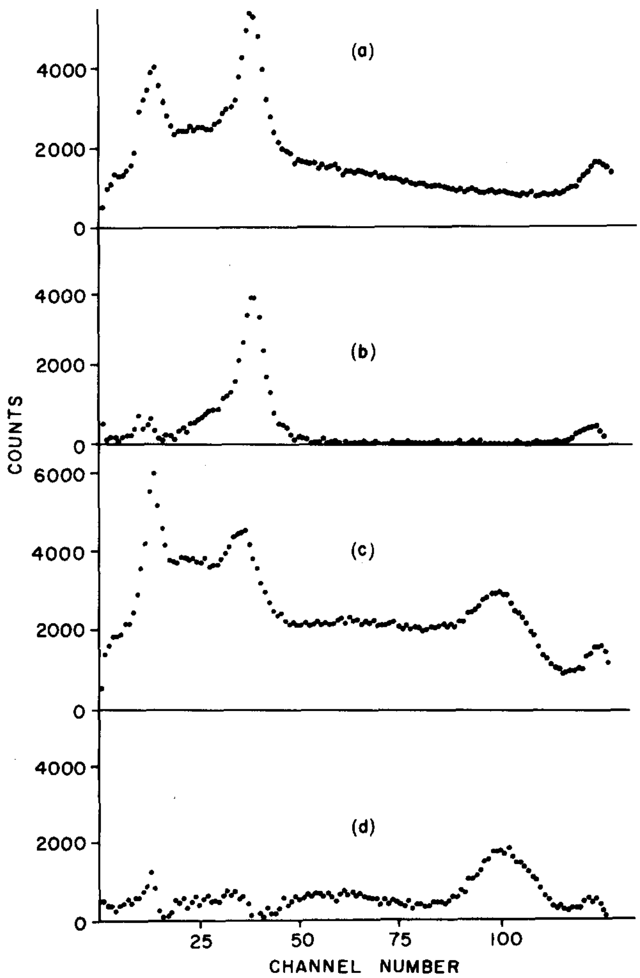

Fig. 2. Energy spectra obtained while scattering 662-keV γ rays from a Cu scattering ring. Spectra (a) and (c) are the raw data for scattering at 140° and 40° while (b) and (d) are for the same pair of angles after background subtraction.

$$\Delta E \simeq \sqrt{\frac{E_s^4 \sin^2\theta(\Delta\theta)^2}{m_0c^4} + \frac{m_0c^2E_s}{300}} \quad \text{(FWHM)}, \quad (7)$$

where the second term $\sqrt{m_0c^2E_s/300}$ is the resolution of the NaI detector expressed in terms of m_0c^2. The first term in this equation is obtained by differentiating Eq. (3b), and it has a maximum value at 38°. For 32° the observed width of the Compton peak, for $E_i = 662$ keV, was ~55 keV compared to ~45 keV for the Compton term and 30 keV for the detector alone. At high angles the energy width is due mainly to the NaI crystal.

The arrangement of Fig. 1 permits fast experiments with such weak sources that there is no significant radiation hazard. The count rate, which depends slightly on source position, never exceeds 500 cpm with a 50-μCi ^{137}Cs source. Since the electronics could easily tolerate a 20-fold increase in this rate, the experiment could be shortened by this same factor by using a source of 1 mCi for demonstrations by the instructor. The radiation dose rate at a practical minimum distance of 0.25 m is about 0.25 mrem/h with the 50-μCi source, compared to 5 mrem/h with a 1-mCi source.

Some data were obtained with a vertical Ge(Li) spectrometer using the geometry shown in the inset of Fig. 4. The figure is to scale and only Pb shielding was used. Counting times (~ 200 sec) were varied as required by the strength of the available sources which ranged from 5 to 50 μCi.

RESULTS AND DISCUSSION

A. Experiment to show Z independence

Typical spectra obtained when scattering the 662-keV γ rays from a ^{137}Cs source by a Cu scattering ring are shown in Fig. 2. The successive peaks in Fig. 2(c) are, beginning at low energy: (i) a composite peak from K x rays of Pb and W; (ii) a peak due to air scattering at a range of angles near 180°; (iii) the main scatter peak from Cu at an angle of about 40°; and (iv) the full-energy peak due to direct transmission or possibly Rayleigh-scattered in Pb/W. The spectrum in Fig. 2(d) is obtained from Fig. 2(c) by running the analyzer, without the scattering ring, in the subtract mode for an equal time. The spectrum of Fig. 2(b) is obtained from Fig. 2(a) in a similar manner. This identifies clearly, for the student, which peak is due to scattering from the ring and produces scatter peaks of improved symmetry, but at some cost in statistical accuracy and a doubling of experimental time. The centroids of the peaks in Figs. 2(a) and 2(c) may be located as accurately as those in Figs. 2(b) and 2(d), but we feel the didactic advantage of the subtraction is important. Also it should be noted that in Compton's experiment with x rays both an "elastic" and an "inelastic" peak were observed—the elastic peak is the one involved in x-ray diffraction processes and it illustrates that the electron is tied to a massive object. Since our wavelength is much shorter than Compton's (by a factor of about 40) the "Debye–Waller" factor of the scattering materials reduces the elastic peak to a negligible size and only the wavelength shifted peak can be seen.

Figure 3 shows the experimentally determined centroids for scattering at 134° from rings of graphite (C), Al, Ti, Cu, Mo, Cd, and Pb. Each spectrum was acquired in 200 sec with the exception of that for Pb which required 1000 sec because of photoelectric absorption of the scattered radiation and competition between Compton and photoelectric effects in the incident beam. Uncertainties of the centroids are about the size of the experimental points. Nevertheless, we have found that an effective classroom demonstration (during a 50-min class) with rings of four different materials—C, Al, Cu, Pb—can be accomplished with the 50-μCi ^{137}Cs source and 100-sec counting times. The independence of the results on the atomic number may be demonstrated readily. Such results, which would have settled the protracted Duane—Compton controversy[1] if available in 1923, could be obtained in about 30 min using a 1-mCi source, with better quality than in Fig. 3.

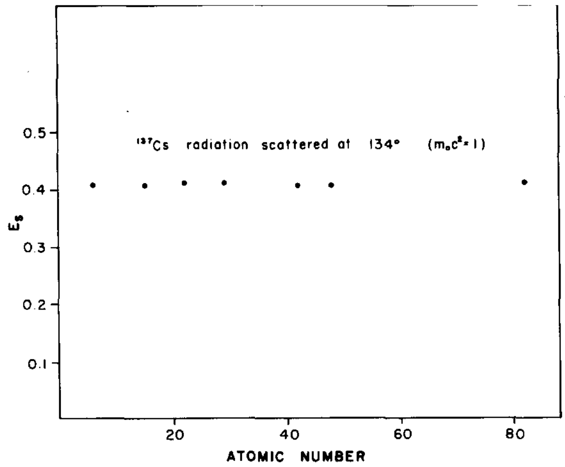

Fig. 3. Energy in units of m_0c^2 of the centroid of the peak for 662-keV γ rays scattered at 134° for scattering rings of elements with atomic numbers from 6 to 82.

B. Experiment to verify relativistic dynamics

We have used this experiment in a laboratory course (for students beginning the second year of a four-year physics program), and found that the student can complete the experimental work in about 90 min. First, he calibrates the multichannel analyzer, and to do this the shielding in Fig. 1 is removed and a ¼-in. lead plate is placed over the detector. A very short run then gives two peaks—a full energy 662-keV peak and a lead x-ray peak at an effective mean energy of 74 keV. Assuming the system is linear an energy calibration is obtained. Next, he replaces the shielding to the arrangement shown in Fig. 1 and tests the subtract mode of the analyser with a Cu ring in place for both add and subtract. For 2 min in each mode a convincing demonstration is obtained, as the spectrum completely vanishes except for statistical fluctuations.

It is then necessary to take data for several positions of the source in Fig. 1. The scale attached to the source rod is normalized by noting the reading when the source is in the plane of the ring (i.e., $\theta = 90°$). Then the student need only note the distance moved, above or below this plane, to evaluate the angle of scatter from the dimensions of the ring—this is simplified if the ring is situated vertically over the aperture on the detector. Data is taken again for 2 min in the add and subtract mode, but this time removing the copper ring for subtraction. Results similar to those shown in Figs. 2(b) and 2(d) are obtained on the analyzer display and the student notes down the peak position on the screen. The students were told that several angles should be taken,

Table I. Data from experiment to test Eqs. (3a) and (3b). (a) Typical results obtained by second-year student using 662-keV γ's and (b) data for 166-keV radiation and 17.44-keV radiation.[d]

	$\theta°$	E_s(keV)	$m_0c^2\left(\frac{1}{E_s}-\frac{1}{E_i}\right)$	$\frac{(E_i-E_s)^2}{2E_iE_s}$	$\cos\theta$	Left-hand side of (3a)	Left-hand side of (3b)[a]
	11.4	650	0.01	0	0.98	0.99	0.99
	40.0	499	0.25	0.04	0.77	0.98	1.02
(a)	57.6	403	0.50	0.13	0.54	0.91	1.04
	92.2	281	1.04	0.39	−0.04	0.61	1.00
	119.2	223	1.52	0.65	−0.49	0.38	1.03
	140.2	177[c]	2.11	1.00	−0.77	0.35	1.34
	140.2	197[b]	1.82	0.83	−0.77	0.22	1.06

$\theta°$	E_i(keV)	E_s(keV)	$m_0c^2\left(\frac{1}{E_s}-\frac{1}{E_i}\right)$	$\frac{(E_i-E_s)^2}{2E_iE_s}$	$\cos\theta$	Left-hand side of (3a)	Left-hand side of (3b)
48	166	150	0.33	0.005	0.67	0.99	1.00
70	166	137	0.65	0.019	0.34	0.98	0.99
83	166	129	0.88	0.032	0.12	0.97	1.09
103	166	119	1.22	0.056	−0.23	0.94	0.99
120	166	112	1.49	0.078	−0.50	0.91	0.99
149	166	104	1.84	0.111	−0.86	0.87	0.98
45	17.44	17.27	0.28	0.5×10^{-4}	0.71	0.99	0.99
90	17.44	16.87	0.99	5×10^{-4}	0	0.99	0.99
135	17.44	16.48	1.70	15×10^{-4}	−0.71	0.99	0.99

[a] The rms deviation of the final column is 0.03. [b] Probable result for final line. [c] Result reported. [d] Taken from paper by A. H. Compton (8).

but left to decide the number for themselves. Most recorded data for six angles. They were then asked to evaluate and tabulate the terms in Eqs. (3a) and 3(b) and discuss their results. Some students concluded that the error on the data was related to the deviation of the left-hand side of Eq. (3a) from unity. Most students did not observe that the angles should be chosen to make $(E_i - E_s)^2/2E_iE_s$ large in order to test the relativistic equation (3b). Points such as these were brought out in subsequent discussion of the results. An example of the data obtained by these students is given in Table I(a), from which it can be seen for angles greater than 60°, Eqs. (3a) and (3b) can be separated. The final line is given both as reported by the student and as it probably should have been observed.

Also shown in Table I are data obtained by us for 166-keV ^{139}Ce radiation and for the data obtained by Compton.[8] The independence of the quantity $m_0c^2(E_s^{-1} - E_i^{-1})$ on incident energy for any given angle is strikingly demonstrated, and validity of classical mechanics for energies less than 100 keV is clear.[6,7]

C. Test of the relativistic energy–momentum relation

The data from these experiments can be used to obtain the energy and momentum of the scattered electron and hence distinguish between Eqs. (2a) and (2b). However, to do this satisfactorily the errors apparent in Table I should be reduced. This is illustrated in Fig. 4, where data (solid circles) obtained from the ^{137}Cs (663 keV) source and an aluminum ring are plotted. These data were obtained by the method described in Sec. B above and, while showing better agreement with the relativistic than the nonrelativistic predictions, are not satisfactory. The error at low angles is large probably because $dK/d\theta$ is large, and the determination of the mean scattering angle from the center of the ring is probably inadequate.

An improved experimental arrangement was used for this test. A Ge(Li) spectrometer was employed as the detector d of the inset to Fig. 4. The source was placed at b and a fixed copper plate at a to give a scattering angle of 160°.

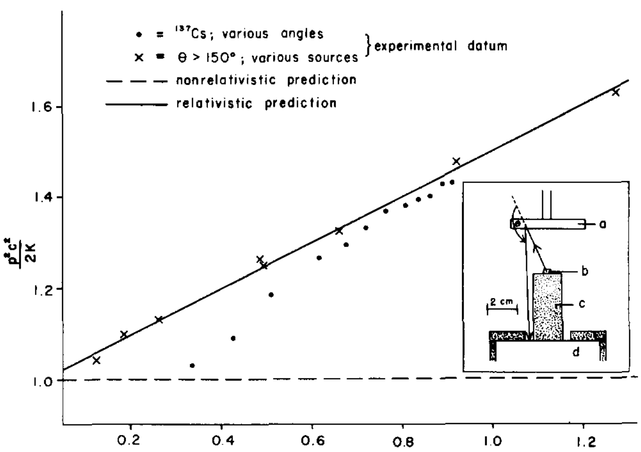

Fig. 4. Relationship between momentum and energy transfer in collisions of photons with free electrons. The dots are data obtained with the apparatus of Fig. 1 when scattering ^{137}Cs γ rays from Al. The crosses are obtained when scattering γ rays of various energies at a fixed angle from an Al plate using a Ge(Li) spectrometer with shielding as shown in the inset. The scattering plate a, source b, and lead shielding c are supported above a vertically mounted Ge(Li) detector of ~ 10% efficiency. Both axes are in units of m_0c^2.

A number of different sources were used including γ rays of 847 (^{56}Mn), 662 (^{137}Cs), 511 (^{22}Na), 477 (^{7}Be), 412 (^{198}Au), 265 (^{75}Ge), 213 (^{75}Ge), and 166 keV (^{139}Ce). These data are plotted as crosses in Fig. 4, and it is seen that they satisfy Eq. (2b) quite well.

D. Rest mass of the electron

The same experimental data obtained with the Ge(Li) spectrometer may be used to test the predictions of Eqs. (5) and (6). As expected from Fig. 4, good agreement is obtained. A simple unweighted average of the experimental points gives an "unreasonably" accurate value of 9.11×10^{-31} kg for m_0 from Eq. (5). The relativistic mass, Eq. (6), exceeds m_0 by a factor of 2 for 662-keV γ rays. Provided that a suitable range of sources and a modest Ge(Li) spectrometer is available—the resolution demands are minimal and the shielding arrangements very simple—this subsidiary experiment gives an excellent test of special relativity which can be completed readily in one laboratory session.

SUMMARY

We have shown that several undergraduate experiments in special relativity may be conducted using Compton scattering of γ rays. The apparatus can be found in most physics laboratories and the counting times with weak sources are short enough for successful classroom demonstrations. The underlying ideas are similar enough to those used in classical mechanics that these experiments can be adopted for courses given in the second year at a North American university.

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[3] A. Pais, Rev. Mod. Phys. **51**, 863 (1979).

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[5] J. A. Gray, Philos. Mag. **26**, 611 (1913); J. Franklin Inst. **190**, 633 (1920).

[6] It is worth noting that while Eq. (3b) yields the oft-quoted result that the wavelength change is independent of incident wavelength, the classical equation (3a) does not do so. This may be seen from Table I where the wavelength change is proportional to $m_0c^2(E_s^{-1} - E_i^{-1})$ and the deviation from this quantity in classical physics is proportional to $(E_i - E_s)^2/2E_iE_s$.

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