where σ_d is the average distance that any single electron moves. Therefore,

$$\sigma_V = \frac{e}{L}\sqrt{N}\frac{\sigma_d}{t_0}R. \tag{3.17}$$

Now N is the total number of conduction electrons in the resistor times the number of walks in time t_0, so

$$N = (nAL) \times \frac{t_0}{\tau} = \frac{nALt_0}{\tau},$$

where n is the number density of conduction electrons and τ is the time between collisions of a *single* electron. The fluctuation in the motion of a single electron is

$$\sigma_d^2 = \langle d^2 \rangle = \langle v_x^2 \tau^2 \rangle = \langle v_x^2 \rangle \tau^2,$$

and this is what we connect to temperature by $\langle E \rangle = \frac{1}{2}m\langle v_x^2 \rangle = \frac{1}{2}kT$, where m is the mass of an electron and we note that motion is only in one dimension. The factor k is Boltzmann's constant which defines the fundamental relationship between temperature and internal energy. Therefore

$$\sigma_d^2 = \frac{kT\tau^2}{m}.$$

We note that (see Eq. (2.14))

$$\frac{L}{A}\frac{2m}{ne^2\tau} = \frac{L}{A}\rho = R,$$

where ρ is the resistivity.[5]

Finally, put this all into Eq. (3.17) to get

$$\begin{aligned}
\sigma_V^2 &= \frac{e^2}{L^2}N\frac{\sigma_d^2}{t_0^2}R^2 \\
&= \frac{e^2}{L^2}\frac{nALt_0}{\tau}\frac{kT\tau^2}{mt_0^2}R^2 \\
&= \frac{A}{L}\frac{ne^2\tau}{m}\frac{kT}{t_0}R^2,
\end{aligned}$$

[5]The definition of τ used here differs from that used in Section 2.2 by a factor of 2. That is because we are dealing with a single electron.

or

$$\langle V^2 \rangle = \frac{2kTR}{t_0}. \tag{3.18}$$

It is customary, however, to express the noise using the equivalent bandwidth $\Delta\nu = 1/2t_0$. Therefore, we have

$$\langle V^2 \rangle = 4kTR\Delta\nu. \tag{3.19}$$

In order to measure the voltage V, we will need to amplify or at least process the signal in some way. Let $g(\nu)$ be the gain of this processing circuit at frequency ν. Then the output voltage fluctuation $d\langle V^2 \rangle$ integrated over some small frequency range $d\nu$ is given by

$$d\langle V^2 \rangle = 4kTRg^2(\nu)\,d\nu.$$

Measurements are made by integrating the signal over a relatively large bandwidth $\Delta\nu$. This bandwidth is typically determined by the gain function $g(\nu)$, which is large only over some finite frequency range. We therefore obtain the expression

$$\langle V^2 \rangle = 4kTRG^2\Delta\nu, \tag{3.20}$$

where G and $\Delta\nu$ are constants defined by

$$G^2\Delta\nu \equiv \int_0^\infty g^2(\nu)\,d\nu. \tag{3.21}$$

3.6.2. Measurements

We will measure the Johnson noise in a series of resistors, and use the result to determine a value for Boltzmann's constant k.

The setup is shown schematically in Fig. 3.22. The voltage across the resistor R is immediately processed by an "amplifier," which essentially multiplies this voltage by a function $g(\nu)$. The output of the amplifier is measured using a digital oscilloscope. You will use the oscilloscope to measure $\langle V^2 \rangle$, given by Eq. (3.20). By changing the value of R (simply by changing resistors), you measure $\langle V^2 \rangle$ as a function of R, and the result should be a straight line. The slope of the line is just $4kTG^2\Delta\nu$, so once you have calibrated the gain function of the amplifier, you can get k. (You can assume the resistor is at room temperature.)

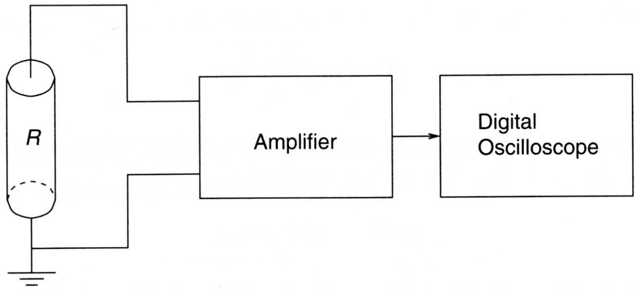

FIGURE 3.22 Schematic for measuring Johnson noise.

Let's look a little more carefully at the properties of the amplifier. We will be working in the several tens of kilohertz range, so to estimate the gain we need, take a bandwidth $\Delta\nu = 10$ kHz. The digital oscilloscope cannot make measurements much smaller than around 0.5 mV, so Eq. (3.20) implies that the nominal gain G must be on the order of 1200 or more to measure the noise in a 1-kΩ resistor. The amplifier also needs to have low noise and good stability itself, if we are going to use it on such a small signal. A high-gain opamp with negative feedback (see Section 3.5) sounds like the right solution.

The bandwidth of the amplifier also needs to be considered. In fact, if we are going to do the job right, we want to make sure that all the bandwidth limitations are given by the amplifier, and not by the oscilloscope, for example. That way, we can measure the function $g(\nu)$ of the amplifier stage only. The oscilloscope bandwidth will depend on the timebase used, that is, the time over which the output voltage is averaged and digitized. As long as the oscilloscope's bandwidth is greater than the amplifier's, you will be OK. You ensure this by putting a bandwidth filter on the output of the amplifier. In the beginning, you will use a commercial bandwidth filter with adjustable lower and upper limits.

The first "amplifier" you will use, therefore, is shown in Fig. 3.23. For now the bandwidth filter is just a box with an input and output, and with knobs you can turn. The gain-producing part of the amplifier, on the other hand, is essentially a cut-and-dry application of opamps and negative feedback. In fact, as shown in Fig. 3.23, two such negative feedback loops are cascaded to get the appropriate gain and input characteristics. The first loop uses a HA5170 opamp and a low gain, while the second stage is higher

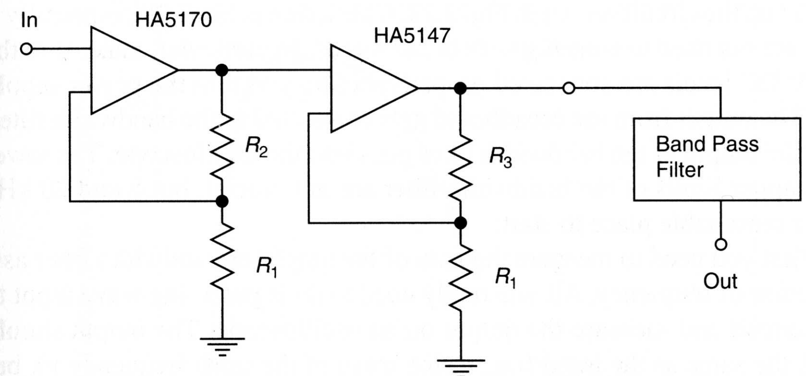

FIGURE 3.23 Amplifier stage for measurement of Johnson noise.

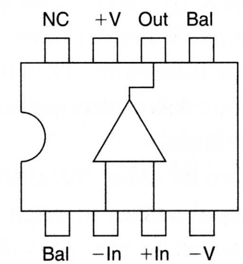

FIGURE 3.24 Pinout diagram for the opamp chips used in this experiment. We are not using the "Bal" connections. The notation "NC" means "no connection."

gain and uses a HA5147.[6] Good starting values to use are $R_1 = 10\,\Omega$, $R_2 = 100\,\Omega$, and $R_3 = 2.2\,\mathrm{k}\Omega$. This gives the first stage a gain of 11 and the second stage a gain of 221 times the bandwidth function imposed by the opamps and the bandwidth filter.

All of these components, including your input resistor R (but not the commercial bandwidth filter), are mounted on a breadboard so you can change things easily. The pinout diagram for the HA5170 and HA5147 is shown in Fig. 3.24. The opamps are powered by ± 12-V levels applied in parallel with 0.1-μF capacitors to ground, to filter off noise in the power supply. Connections to the breadboard are made using wires soldered to BNC connectors.

[6] The credit for figuring out the right opamps and amplifier circuit in general goes to Jeff Fedison, RPI Class of '94. More details on this circuit design are available.

Set up the circuit shown in Fig. 3.23. Check things carefully, especially if you are not used to working with breadboards. In particular, make sure the 12-V DC levels are connected properly, before you turn the power supply on. The output from the breadboard gets connected to the bandwidth filter, and the output of the bandwidth filter goes into the oscilloscope. The lower and upper limits of the bandwidth filter are not crucial, but 5 and 20 kHz are a reasonable place to start.

First you need to measure the gain of the amplifier/bandwidth filter as a function of frequency. All you really need to do is put a sine wave input to the circuit and measure the output on an oscilloscope. The output should look the same as the input (i.e., a sine wave of the same frequency ν), but the amplitude should be bigger. The ratio of the output to input amplitudes is just the gain $g(\nu)$. There is a problem, though. You have built an amplifier of very large gain, around 2.4×10^3, and the output amplitude must be less than a few volts so the opamps do not saturate. That means that the input must be less than a couple of millivolts. That is barely enough to see on an oscilloscope, assuming your waveform generator can make a good sine wave with such a small amplitude.

You get around this problem by using the schematic shown in Fig. 3.25. The waveform generator output passes through a voltage divider, cutting the amplitude down by a known factor. This divided voltage is used as input to the amplifier. It is a good idea to measure the resistor values R_{big} and R_{small} using an ohmmeter, rather than to trust the color code (which can be off by up to 10%). Pick resistors that give you a divider ratio somewhere between 10 and 100. It is also a good idea to see the output of the waveform generator and look at it on the oscilloscope along with the amplifier/bandwidth filter output.

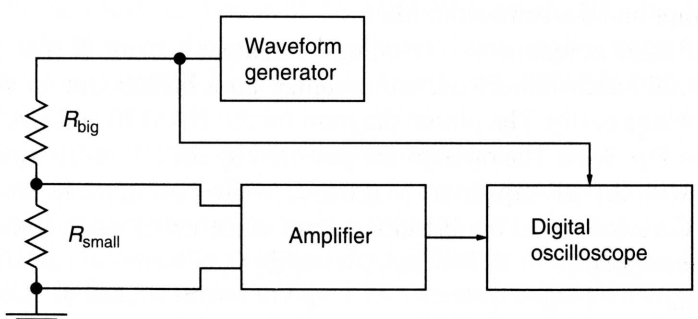

FIGURE 3.25 Calibration scheme for the noise amplifier.

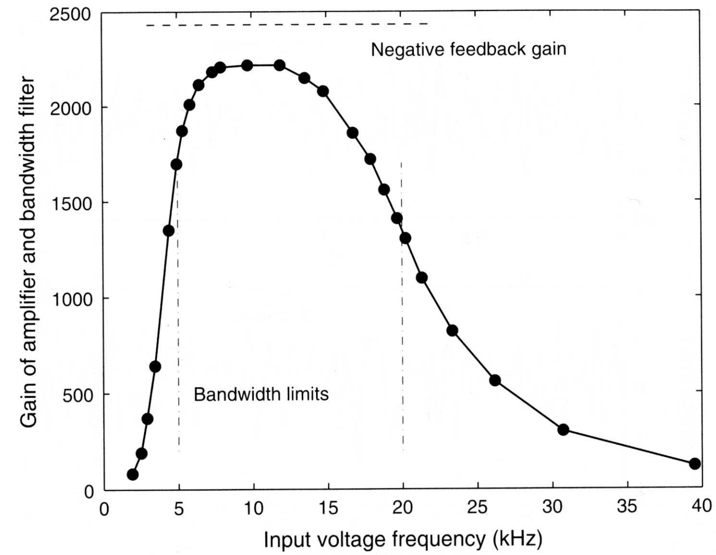

FIGURE 3.26 Sample of data used to determine $g(\nu)$ for the amplifier followed by the commercial bandwidth filter. The simple negative feedback formula gives a gain of 2431, and the bandwidth filter is set for $\nu_{\mathrm{LO}} = 5$ kHz and $\nu_{\mathrm{HI}} = 20$ kHz.

Make your measurements of $g(\nu)$ by varying the frequency of the waveform generator, and recording the output amplitude. Of course, you must also record the input (i.e., generator) amplitude, but if you check it every time you change ν, you can be sure it does not change during your measurement. Measure over a range of frequencies that allows you to clearly see the cutoffs from the bandwidth filter, including the shape as g approaches zero. Also make sure you confirm that the gain is relatively flat in between the limits. An example is shown in Fig. 3.26. The setup used $R_1 = 10\,\Omega$, $R_2 = 100\,\Omega$, and $R_3 = 2.2\,\mathrm{k}\Omega$, so the total gain should be 2431, and with bandwidth filter limits at 5 and 20 kHz. The main features seem to be correct, although the filter has apparently decreased the maximum gain a bit.

Now take measurements of the actual Johnson noise as a function of R. Remove the waveform generator and voltage divider inputs, and put the resistor you want to measure across the input to the amplifier. Set the time per division on the oscilloscope so that its bandwidth limit is much larger than the upper frequency you used on the bandwidth filter. For example, if there are 10,000 points (i.e., samples) per trace and you set the scope to

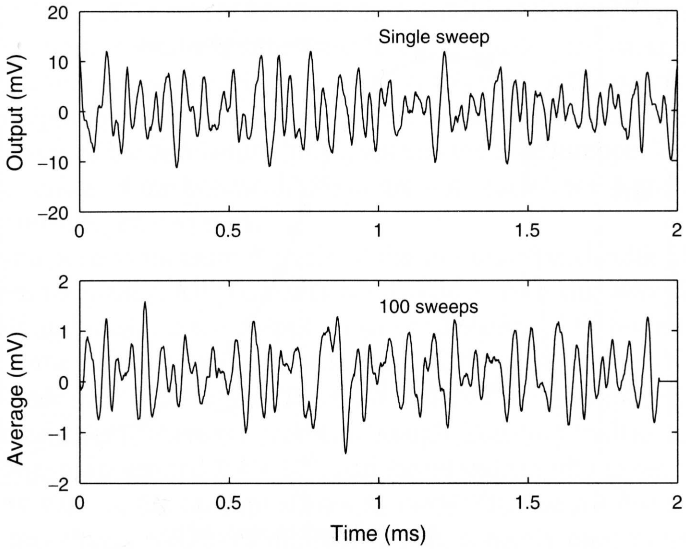

FIGURE 3.27 Oscilloscope traces of the output of the bandwidth filter, and for 100 traces averaged together by the oscilloscope. Note the difference in the vertical scales.

0.2 ms/div, then the time per sample is 0.2 μs since there are ten divisions. The bandwidth is the reciprocal of twice this time or 2.5 MHz. If the filter cuts off at 20 kHz, then this would be fine.

Use resistors with R near zero (10 Ω) and up to $R \approx 10\,\text{k}\Omega$. The oscilloscope trace will look like an oscillatory signal, but that is because you are (likely) using tight bandwidth limits. What would the trace look like if the lower limit was only slightly smaller than the upper limit?

Figure 3.27 shows a single sweep trace on the scope directly from the output of the bandwidth filter, and the average (as done by the scope itself) of 100 traces. The average looks the "same" as the single sweep, but it is 10 times smaller. (Note the difference in the vertical scales.) It is clear, therefore, that the oscillations in the input signal are random in phase, even though they are confined within the limits of the bandwidth filter. Most digital oscilloscopes have the ability to calculate and display for you the mean and variance of the trace. This will be useful for your analysis.

You need to determine the value of $G^2\Delta\nu = \int_0^\infty g^2(\nu)\,d\nu$. Make a plot of $g^2(\nu)$ as a function of ν and estimate the integral under the curve. You can try to estimate this graphically, but you can easily get an accurate answer using the MATLAB function `trapz`, which performs a trapezoidal

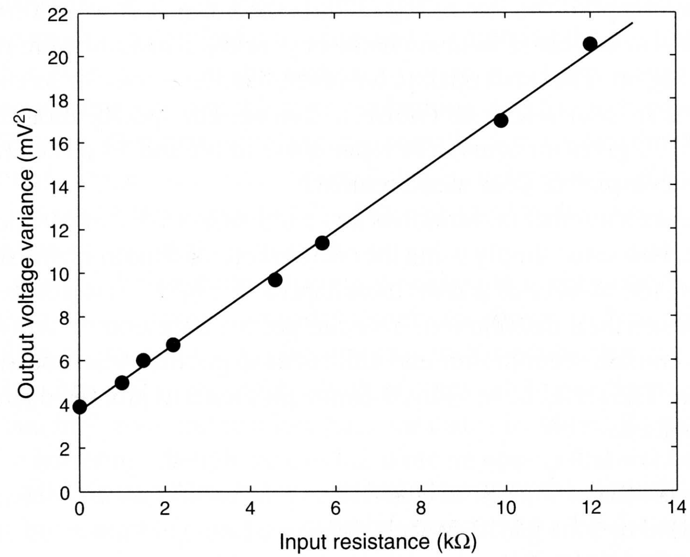

FIGURE 3.28 Data taken by measuring the standard deviation of the output voltage signal, as a function of the input resistor value. The slope gives k, while the intercept gives the equivalent input noise voltage, after correcting for the amplifier gain × bandwidth.

integration given a list of (x, y) values. For the data of Fig. 3.26 one finds that

$$G^2\Delta\nu = (7.9 \pm 0.5) \times 10^7 \text{ kHz}.$$

Next we make a plot of $\langle (V - \langle V\rangle)^2\rangle$ as a function of R. Note that since $\langle V\rangle = 0$, the above expression reduces to $\langle V^2\rangle$. The plot is shown in Fig. 3.28 and a linear fit gives

$$\langle V^2\rangle/R = (1.33 \pm 0.08)\ \text{mV}^2/\text{k}\Omega$$

and an intercept at 4 mV^2.

We can now calculate Boltzmann's constant k from the above data using Eq. (3.20) and setting $T = 298$ K (room temperature). Using units of hertz, volts, and ohms, we write

$$k = \frac{\langle V^2\rangle/R}{4TG^2\Delta\nu} = \frac{(1.33\pm0.08)\times10^{-9}}{4\times298\times(7.9\pm0.5)\times10^{10}} = (1.42\pm0.13)\times10^{-23}\ \text{J/K}.$$

This result is in excellent agreement with the accepted value $k = 1.38 \times 10^{-23}$ J/K.

The intercept of the line in Fig. 3.28 is the noise at $R = 0$. You would expect this to be zero if Johnson noise in your input resistor were the only thing going on. The input opamp, however, has some noise of its own, due to internal Johnson noise, shot noise, and so on. The specification sheet for the HA5170 gives an equivalent input noise of around 10 nV/$\sqrt{\text{Hz}}$. How does this compare to your measurement?

There are a number of variations and extensions to this experiment. For example, instead of simply using the oscilloscope to determine the standard deviation, use MATLAB and the trace data (as in Fig. 3.27) to get the values and examine their distribution. You can get the data into an array `trace`, and you can use `mean(trace)` and `std(trace)` to get the mean and standard deviation. The series of MATLAB commands used to plot the distribution might look like

```
bins = linspace(min(trace), max(trace), 50);
[n, x] = hist(trace, bins);
stairs(x, n);
```

The single sweep trace in Fig. 3.27 is plotted this way in Fig. 3.29. The distribution is rather Gaussian-like, as you expect, but you could test to

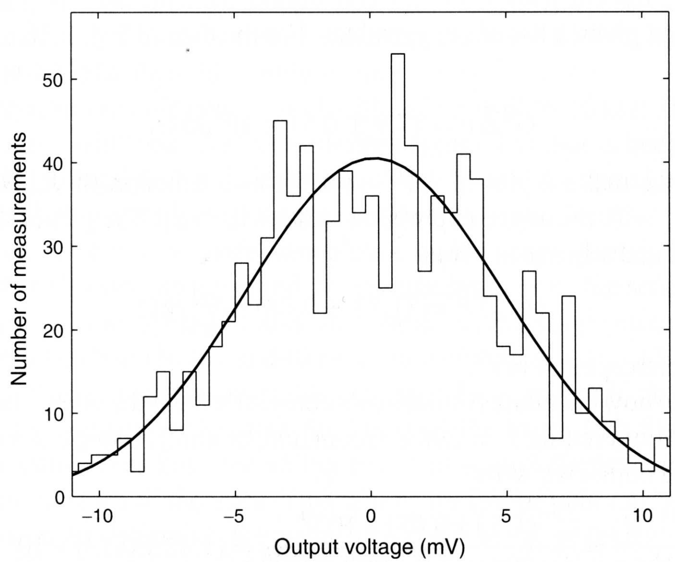

FIGURE 3.29 Histogram of the individual voltage values from a single sweep trace. The line is a Gaussian distribution, with the mean and standard deviation determined from the trace data, and normalized to the number of measurements.

see whether this is really the case by comparing it to the Gaussian with the same mean and standard deviation, and considering the χ^2. (See Chapter 10 for definitions and discussions of these quantities.) Some digital oscilloscopes have the capability of performing a real-time Fourier analysis of the input. That means that you can actually demonstrate that the noise spectrum $d\langle V^2\rangle/d\nu$ is indeed "white," that is, independent of frequency. This is straightforward data to take, but will require that you learn more about Fourier analysis to interpret it.

One nontrivial circuit modification would be to make your own bandwidth filter. For example, consider the circuit shown in Fig. 3.12.[7] Try assembling components that give you reasonable parameters for the gain integral in Eq. (3.21). A simpler kind of filter might simply be two RC filters, one high pass and one low pass, cascaded in series. If you want to do active buffering, though, be careful to use an opamp that works at these frequencies. Another interesting variation is to use a few-kiloohm resistor as input, but something mechanically large and strong enough to take some real temperature change. If you immerse the resistor in liquid nitrogen, for example, it should make a large (and predictable) change in the Johnson noise.

3.7. CHAOS

We now discuss a measurement that uses nonlinear electronic components to explore phenomena characteristic of complex physical systems.

3.7.1. The Logistic Map and Frequency Bifurcations

We are used to the notion that physical systems are described by differential equations that can be exactly solved for all times, given an appropriate set of initial conditions. This is not true in complex systems governed by nonlinear equations. A typical example is the flow of fluids. At low velocity one can identify individual "streamlines" and predict their evolution. However, when a particular combination of velocity, viscosity, and boundary dimensions is reached, *turbulence* sets in and eddies and vortices are formed. The motion becomes *chaotic*. Many chaotic systems exhibit self-similarity: that

[7]This, in fact, is what Johnson used in his 1928 paper. You might want to look it up, and compare your results to his.